Chapter 16

Rational Functions: Local Analysis Near \( \infty \)

Local I-O Rule Near \( \infty \), 271 – Height-sign Near \( \infty \), 274 – Slope-sign Near \( \infty \), 276 – Concavity-sign Near \( \infty \), 278 – Local Graph Near \( \infty \), 282.

To do local analysis we work in a neighborhood of some given input and thus count inputs from the given input since it is the center of the neighborhood. When the given input is \( \infty \), counting from \( \infty \) means setting \( x \leftarrow \text{large} \) and computing with powers of \( \text{large} \) in descending order of sizes.

Recall that the principal term near \( \infty \) of a given polynomial function \( POLY \) is simply its highest power term which is therefore easy to extract from the global input-output rule. The approximate input-output rule near \( \infty \) of \( POLY \) is then of the form

\[
POLY(x) \bigg|_{x \text{ near } \infty} = \text{Highest Term } POLY + [...] 
\]

However, the complication here is that to get the principal part near \( \infty \) of a rational function we must approximate the two polynomial and divide—or the other way round—and the result need not be a polynomial but can also be a negative-exponent power function and the main issue will be whether to do the approximation before or after the division.

16.1 Local Input-Output Rule Near \( \infty \)

Given a rational function \( RAT \), we look for the function whose input-output rule will be simpler than the input-output rule of \( RAT \) but whose local graph
CHAPTER 16. RATIONAL FUNCTIONS: LOCAL ANALYSIS NEAR ∞

near ∞ will be qualitatively the same as the local graph near ∞ of RAT.

More precisely, given a rational function RAT specified by the global input-output rule

\[
x \xrightarrow{\text{RAT}} \text{RAT}(x) = \frac{\text{POLY}_{\text{Num}}(x)}{\text{POLY}_{\text{Den}}(x)}
\]

what we will want then is an approximation for the output of the local input-output rule near ∞

\[
x \big|_{x \to \infty} \xrightarrow{\text{RAT}} \text{RAT}(x) \big|_{x \to \infty} = \frac{\text{POLY}_{\text{Num}}(x) \big|_{x \to \infty}}{\text{POLY}_{\text{Den}}(x) \big|_{x \to \infty}}
\]

from which to extract whatever controls the wanted feature.

1. Since the center of the neighborhood is ∞, we localize both
   - \(\text{POLY}_{\text{Num}}(x)\)
   and
   - \(\text{POLY}_{\text{Den}}(x)\)

by writing them in descending order of exponents.

2. Depending on the circumstances, we will take one of the following two routes to extract what controls the wanted feature:
   - The short route to Princ. TERM \(\text{RAT}(x) \big|_{x \to \infty}\), that is:
     i. We approximate both \(\text{POLY}_{\text{Num}}(x) \big|_{x \to \infty}\) and \(\text{POLY}_{\text{Den}}(x) \big|_{x \to \infty}\)
        to their principal term—that is to just their highest size term—
        which, since \(x \to \infty\), is their highest exponent term:

\[
\text{Localize near } \infty \quad \frac{\text{POLY}_{\text{Num}}(x) \big|_{x \to \infty}}{\text{POLY}_{\text{Den}}(x) \big|_{x \to \infty}}
\]

   ii. In order to divide \(\text{Princ. TERM}_{\text{Num}}(x) \big|_{x \to \infty}\), that is the principal term near ∞ of the numerator of RAT by \(\text{Princ. TERM}_{\text{Den}}(x) \big|_{x \to \infty}\),
        that is the principal term near ∞ of the denominator of RAT we use monomial division

\[
\frac{ax^m}{bx^n} = \frac{a}{b}x^{m-n} = \text{x near } \infty
\]

where \(+m \ominus +n\) can turn out positive, negative or 0

\[
\text{Princ. TERM } \text{RAT}(x) \big|_{x \to \infty} = \frac{\text{Princ. TERM}_{\text{Num}}(x) \big|_{x \to \infty}}{\text{Princ. TERM}_{\text{Den}}(x) \big|_{x \to \infty}}
\]
16.1. LOCAL I-O RULE NEAR $\infty$

\[
\frac{\text{coeff. Princ. TERM}_{\text{Num}}(x)}{\text{coeff. Princ. TERM}_{\text{Den}}(x)} \cdot x^\text{UppDeg. POLY}_{\text{Num}}(x) - \text{UppDeg. POLY}_{\text{Den}}(x)
\]

\[
\frac{\text{coeff. Princ. TERM}_{\text{Num}}(x)}{\text{coeff. Princ. TERM}_{\text{Den}}(x)} \cdot x^\text{RatDeg. RAT}(x)
\]

The resulting monomial is $\text{Princ. TERM RAT}(x)\big|_{x \to \infty}$, that is the principal term of the rational function $\text{RAT}$ near $\infty$:

\[
\begin{array}{c}
\text{POLY}_{\text{Num}}(x) \quad \text{Localize near } \infty \\
\text{POLY}_{\text{Den}}(x) \quad \text{Localize near } \infty
\end{array}
\]

\[
\begin{array}{c}
\text{POLY}_{\text{Num}}(x) \big|_{x \to \infty} \quad \text{i. Approximate} \\
\text{POLY}_{\text{Den}}(x) \big|_{x \to \infty} \quad \text{i. Approximate}
\end{array}
\]

\[
\begin{array}{c}
\text{Princ. TERM}_{\text{Num}}(x) \big|_{x \to \infty} + [\ldots] \\
\text{Princ. TERM}_{\text{Den}}(x) \big|_{x \to \infty} + [\ldots]
\end{array}
\]

\[
\text{Princ. TERM RAT}(x) \big|_{x \to \infty} + [\ldots]
\]

- The long route to $\text{Princ. PART RAT}(x)\big|_{x \to \infty}$:
  
  i. In order to divide $\text{POLY}_{\text{Num}}(x)\big|_{x \to \infty}$ by $\text{POLY}_{\text{Den}}(x)\big|_{x \to \infty}$, we set up the division as a long division, that is $\text{POLY}_{\text{Den}}(x)\big|_{x \to \infty}$ dividing into $\text{POLY}_{\text{Num}}(x)\big|_{x \to \infty}$:

\[
\begin{array}{c}
\text{POLY}_{\text{Num}}(x) \quad \text{Localize near } \infty \\
\text{POLY}_{\text{Den}}(x) \quad \text{Localize near } \infty
\end{array}
\]

\[
\begin{array}{c}
\text{POLY}_{\text{Num}}(x) \big|_{x \to \infty} \\
\text{POLY}_{\text{Den}}(x) \big|_{x \to \infty}
\end{array}
\]

\[
\text{Princ. TERM}_{\text{Num}}(x) \big|_{x \to \infty}
\]

ii. We approximate by stopping the long division as soon as we have the principal part that has the feature(s) we want:

\[
\begin{array}{c}
\text{POLY}_{\text{Den}}(x) \big|_{x \to \infty} \\
\text{POLY}_{\text{Num}}(x) \big|_{x \to \infty}
\end{array}
\]

\[
\text{Princ. PART RAT}(x) \big|_{x \to \infty} + [\ldots]
\]

\[
\text{POLY}_{\text{Den}}(x) \big|_{x \to \infty} \\
\text{POLY}_{\text{Num}}(x) \big|_{x \to \infty}
\]

\[
\text{Princ. PART RAT}(x) \big|_{x \to \infty} + [\ldots]
\]
3. Which route we will take in each particular case will depend both on the wanted feature(s) near \( \infty \) and on the rational degree of \( RAT \) and so we will now look separately at how we get \( \text{Height-sign}\big|_{x \to \infty}, \text{Slope-sign}\big|_{x \to \infty} \) and \( \text{Concavity-sign}\big|_{x \to \infty} \)

**Local Analysis near \( \infty \)**

When the wanted features are to be found near \( \infty \), the rational degree of the rational function tells us up front whether or not the short route will allow us to extract the term that controls the wanted feature.

### 16.2 Height-sign Near \( \infty \)

No matter what the rational degree of the given rational function \( RAT \), \( \text{Princ.TERM RAT}(x)\big|_{x \to \infty} \) will give us \( \text{Height-sign}\big|_{x \to \infty} \) because, no matter what its exponent, any power function has \( \text{Height-sign}\big|_{x \to \infty} \).

So, no matter what the rational degree of \( RAT \), to extract the term responsible for \( \text{Height-sign}\big|_{x \to \infty} \) we can take the short route to \( \text{Princ.TERM RAT}(x)\big|_{x \to \infty} \):

\[
\begin{align*}
\text{PolyNum}(x) & \quad \text{Localize near } \infty \\
\text{PolyDen}(x) & \quad \text{Localize near } \infty \\
\text{PolyNum}(x)\big|_{x \to \infty} & \quad \text{I. Approximate} \\
\text{PolyDen}(x)\big|_{x \to \infty} & \quad \text{I. Approximate} \\
\text{Princ.TERMNum}(x)\big|_{x \to \infty} & \quad + [\ldots] \\
\text{Princ.TERMDen}(x)\big|_{x \to \infty} & \quad + [\ldots] \\
\text{Princ.TERM RAT}(x)\big|_{x \to \infty} & \quad + [\ldots]
\end{align*}
\]

**Example 1.** Given the rational function \( DOUGH \) specified by the global input-output rule

\[
x \rightarrow DOUGH(x) = \frac{+12x^5 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6}
\]

find \( \text{Height-sign } DOUGH\big|_{x \to \infty} \).

a. We localize both the numerator and the denominator near \( \infty \)—which amounts only to making sure that the terms are in descending order of exponents.

\[
\begin{align*}
\text{PolyNum}(x) & \quad \text{Localize near } \infty \\
\text{PolyDen}(x) & \quad \text{Localize near } \infty \\
\text{PolyNum}(x)\big|_{x \to \infty} & \quad +[\ldots] \\
\text{PolyDen}(x)\big|_{x \to \infty} & \quad +[\ldots]
\end{align*}
\]

b. Inasmuch as \( \text{Princ.TERM DOUGH}(x)\big|_{x \to \infty} \) has \( \text{Height} \) no matter what the degree, in order to extract the term that controls \( \text{Height-sign}\big|_{x \to \infty} \) we take the short route to \( \text{Princ.TERM DOUGH(x)}\big|_{x \to \infty} \):
i. We approximate

\[
\begin{align*}
\frac{+12x^5-6x^3+8x^2+6x-9}{-3x^2-5x+6} &\approx \frac{+12x^5}{-3x^2} + [\ldots] \\
\frac{+12x^5-6x^3+8x^2+6x-9}{-3x^2-5x+6} &\approx \frac{+12x^5}{-3x^2} + [\ldots]
\end{align*}
\]

that is we approximate

- the numerator \(+12x^5 - 6x^3 + 8x^2 + 6x - 9\) to its principal term, \(-12x^5\)
- the denominator \(-3x^2 - 5x + 6\) to its principal term, \(-3x^2\)

ii. And then we divide:

\[
\begin{align*}
\frac{+12x^5-6x^3+8x^2+6x-9}{-3x^2-5x+6} &\approx \frac{+12x^5}{-3x^2} + [\ldots] \\
\frac{+12x^5-6x^3+8x^2+6x-9}{-3x^2-5x+6} &\approx \frac{+12x^5}{-3x^2} + [\ldots]
\end{align*}
\]

where

\[
\frac{+12x^5}{-3x^2} = \frac{+12 \cdot x \cdot x \cdot x \cdot x \cdot x}{-3 \cdot x \cdot x} = \frac{-12}{3} x^5 - 2
\]

The more usual way to write all this is something as follows:

\[x \mid_{x \text{ near } \infty} \rightarrow \text{DOUGH} \rightarrow \text{DOUGH}(x) \mid_{x \text{ near } \infty} = \frac{+12x^5 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6} \mid_{x \text{ near } \infty} = \frac{+12x^5 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6} \mid_{x \text{ near } \infty} = \frac{+12x^5 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6} \mid_{x \text{ near } \infty} = \frac{+12x^5 + [\ldots]}{-3x^2 + [\ldots]} = \frac{-12}{3} x^5 - 2 + [\ldots]
\]

Whatever we write, the principal term of \text{DOUGH} \text{ near } \infty \text{ is } \frac{-12}{3} x^3 \text{ and it gives }

\[Height\text{-sign } \text{DOUGH} \mid_{x \text{ near } \infty} = (-, +)
\]

**Example 2.** Given the function \(PAC\) specified by the global input-output rule

\[x \xrightarrow{PAC} PAC(x) = \frac{-12x^5 + 7x + 4}{+4x^6 - 6x^4 - 17x^2 - 2x + 10}
\]
find Height-sign $PAC|_{x \nearrow \infty}$.

Inasmuch as $Princ.\ TERM PAC(x) |_{x \nearrow \infty}$ has Height no matter what the degree, in order to extract the term that controls $Height-sign|_{x \nearrow \infty}$ we take the short route to $Princ.\ TERM DOUGH(x) |_{x \nearrow \infty}$:

$$
\frac{x|_{x \nearrow \infty}}{PAC} \rightarrow PAC(x)|_{x \nearrow \infty} = \frac{-12x^3 + 7x + 4}{+4x^5 - 6x^4 - 17x^2 - 2x + 10}|_{x \nearrow \infty} = \frac{-12x^3 + 7x + 4}{+4x^5 - 6x^4 - 17x^2 - 2x + 10}|_{x \nearrow \infty} = \frac{-12}{+4}x^3 + [+\ldots]
$$

and we get that

$$
Height-sign PAC|_{x \nearrow \infty} = (-, -)
$$

### 16.3 Slope-sign Near $\infty$

In the case of $Slope-sign RAT|_{x \nearrow \infty}$, there are two cases depending on the rational degree of the given rational function:

- If the rational function $RAT$ is either:
  - A regular rational function, that is of rational degree $> 1$ or $< 0$
  or
  - An exceptional rational function of rational degree $= 1$
  that is not an exceptional rational function of rational degree $= 0$, then

$$
Princ.\ TERM RAT(x)|_{x \nearrow \infty}
$$

will be a power function that will have $Slope$ near $\infty$ and so in order to extract the term that controls $Slope-sign|_{x \nearrow \infty}$ we take the short route to $Princ.\ TERM RAT(x)|_{x \nearrow \infty}$:

$$
POLY_{Num}(x)|_{x \nearrow \infty} \rightarrow POLY_{Num}(x)|_{x \nearrow \infty} \rightarrow Princ.\ TERM_{Num}(x)|_{x \nearrow \infty} + [\ldots]
$$

$$
POLY_{Den}(x)|_{x \nearrow \infty} \rightarrow POLY_{Den}(x)|_{x \nearrow \infty} \rightarrow Princ.\ TERM_{Den}(x)|_{x \nearrow \infty} + [\ldots]
$$

$$
Princ.\ TERM RAT(x)|_{x \nearrow \infty} + [\ldots]
$$
**Example 3.** Given the rational function \( SOUTH \) specified by the global input-output rule

\[
x \xrightarrow{SOUTH} SOUTH(x) = \frac{-3x^2 - 5x + 6}{+12x^4 - 6x^3 + 8x^2 + 6x - 9}
\]

find Slope-sign of \( SOUTH \) near \( \infty \)

1. We get the local graph near \( \infty \) of \( SOUTH \)

a. We have

\[
x \mid_{x \text{ near } \infty} \longrightarrow \left. SOUTH(x) \right|_{x \text{ near } \infty} = \left. \frac{-3x^2 - 5x + 6}{+12x^4 - 6x^3 + 8x^2 + 6x - 9} \right|_{x \text{ near } \infty}
\]

\[
= \left. \frac{-3x^2 - 5x + 6}{+12x^4 - 6x^3 + 8x^2 + 6x - 9} \right|_{x \text{ near } \infty}
\]

We now proceed with the two steps:

\[
\frac{-3x^2 - 5x + 6}{+12x^4 - 6x^3 + 8x^2 + 6x - 9} \quad \text{Approximate} \quad \frac{-3x^2 + [...]}{+12x^4 + [...]}
\]

\[
\text{Divide} \quad \frac{-3x^2 + [...]}{+12x^4 + [...]}
\]

\[
\text{and then we divide:}
\]

\[
= \frac{-3}{+12} x^2 - 5 + [...] = \frac{-1}{4} x^{-3} + [...]
\]
c. Since the degree of the power function

\[ x^{POWER} \rightarrow POWER(x) = -\frac{1}{4}x^{-3} \]

which approximates \( SOUTH \) near \( \infty \) is < 0, the power function \( POWER \) has all three features, concavity, slope and height. (This was of course to be expected from the fact that the rational degree of \( SOUTH \) is < 0.)

ii. We get

Slope-sign of \( SOUTH \) near \( \infty \) = (/, \)

If the rational function \( RAT \) is an exceptional rational function whose rational degree = 0, then \( \text{Princ. TERM RAT(x) at } x \rightarrow \infty \) will be an exceptional power function with exponent = 0 and \( \text{Princ. TERM RAT(x) at } x \rightarrow \infty \) will not have Slope and so in order to extract the term that controls Slope-sign \( \text{at } x \rightarrow \infty \) we will have to take the long route to a \( \text{Princ. PART RAT(x) at } x \rightarrow \infty \) that has Slope:

\[ \frac{\text{POLY}_{\text{Num}}(x)}{\text{POLY}_{\text{Den}}(x)} \bigg|_{x \rightarrow \infty} \]

16.4 Concavity-sign Near \( \infty \)

In the case of Concavity-sign \( RAT \) at \( x \rightarrow \infty \), there are two cases depending on the rational degree of the given rational function.

- If the rational function \( RAT \) is a regular rational function, that is if the rational degree of \( RAT \) is either > 1 or < 0, then \( \text{Princ. TERM RAT(x) at } x \rightarrow \infty \) will be a regular power function, that is a power function whose exponent is either > 1 or < 0 and then, in either case, \( \text{Princ. TERM RAT(x) at } x \rightarrow \infty \) will have Concavity and so in order to extract the term that controls Concavity-sign \( \text{at } x \rightarrow \infty \) we take the short route to \( \text{Princ. TERM Den(x) at } x \rightarrow \infty \):
16.4. CONCAVITY-SIGN NEAR $\infty$

**Example 4.** Given the rational function $SOUTH$ specified by the global input-output rule

$$x \xrightarrow{SOUTH} SOUTH(x) = \frac{-3x^2 - 5x + 6}{+12x^4 - 6x^3 + 8x^2 + 6x - 9}$$

find Concavity-sign of $SOUTH$ near $\infty$

i. We get the local graph near $\infty$ of $SOUTH$

a. We have

$$x \bigg|_{x \to \infty} \xrightarrow{SOUTH} SOUTH(x) \bigg|_{x \to \infty} = \frac{-3x^2 - 5x + 6}{+12x^4 - 6x^3 + 8x^2 + 6x - 9} \bigg|_{x \to \infty}$$

We now proceed with the two steps:

$$\frac{-3x^2 - 5x + 6}{+12x^4 - 6x^3 + 8x^2 + 6x - 9} \xrightarrow{Approximate} \frac{-3x^2 + [\ldots]}{+12x^4 + [\ldots]}$$

b. The more usual presentation is:

$$x \bigg|_{x \to \infty} \xrightarrow{SOUTH} SOUTH(x) \bigg|_{x \to \infty} = \frac{-3x^2 - 5x + 6}{+12x^4 - 6x^3 + 8x^2 + 6x - 9} \bigg|_{x \to \infty}$$

We approximate $-3x^2 - 5x + 6 \bigg|_{x \to \infty}$ and $+12x^4 - 6x^3 + 8x^2 + 6x - 9 \bigg|_{x \to \infty}$

$$= -\frac{1}{4}x^3 + [\ldots]$$
and then we divide:

\[
\begin{align*}
  = & -3x^2 - 5 + \ldots \\
  = & 4x^{-3} + \ldots 
\end{align*}
\]

c. Since the degree of the power function

\[
x^{POWER} \rightarrow POWER(x) = -\frac{1}{4}x^{-3}
\]

which approximates SOUTH near \(\infty\) is < 0, the power function POWER has all three features, concavity, slope and height. (This was of course to be expected from the fact that the rational degree of SOUTH is < 0.)

ii. We get

\[
\text{Concavity-sign of SOUTH near } \infty = (\cap, \cap)
\]

- If the rational function RAT is an exceptional rational function that is if the rational degree of RAT is either = 1 or = 0 then \(\text{Princ. TERM RAT}(x) \mid_{x \to \infty}\) will be an exceptional power function with exponent either = 1 or = 0 (Chapter 7) and in both cases \(\text{Princ. TERM RAT}(x) \mid_{x \to \infty}\) will not have Concavity and in order extract the term that controls Concavity-sign\(\mid_{x \to \infty}\) we will have to take the long route to a \(\text{Princ. PART RAT}(x) \mid_{x \to \infty}\) that does have Concavity.

**EXAMPLE 5.** Given the rational function BATH specified by the global input-output rule

\[
x \rightarrow BATH(x) = \frac{+x^3 - 5x^2 + x + 6}{+x^2 - 4x + 3}
\]

find Concavity-sign \(BATH\mid_{x \to \infty}\).

a. The localization step is to localize both the numerator and the denominator near \(\infty\)—which amounts only to making sure that the terms are in descending order of exponents.
16.4. CONCAVITY-SIGN NEAR $\infty$

\[ \frac{+x^3-5x^2+x+9}{+x^2-4x+3} \]

Localize near $\infty$

\[ \frac{+x^3-5x^2+x+9|_{x \to \infty}}{+x^2-4x+3|_{x \to \infty}} \]

\[ \begin{align*}
\text{i. Approximate} \\
+1x^2 - 4x + 3 & \quad \text{dividing into} \\
+1x^3 - 5x^2 + x + 9 & \quad \text{principal part of the quotient} \\
\text{that has Concavity:} \\
+1x + \ldots
\end{align*} \]

\[ \begin{align*}
\text{i. Approximate} \\
+1x^2 - 4x + 3 & \quad \text{dividing into} \\
+1x^3 - 5x^2 + x + 9 & \quad \text{principal part of the quotient} \\
\text{that has Concavity:} \\
+1x + \ldots
\end{align*} \]

\[ \begin{align*}
\text{i. Approximate} \\
x - 1 - 6x^{-1} & \quad \text{since it is the term responsible for Concavity.}
\end{align*} \]

The more usual way to write all this is:

b. Since Princ. TERM $BATH(x) |_{x \to \infty}$ has no Concavity, the extraction step to get Concavity-sign $BATH|_{x \to \infty}$ must take the long route to a Princ. PART $BATH(x) |_{x \to \infty}$ that has Concavity:

i. We set up the division as a long division:

\[ \frac{+x^3-5x^2+x+9}{+x^2-4x+3} \]

\[ \begin{align*}
\text{i. Approximate} \\
+1x^2 - 4x + 3 & \quad \text{dividing into} \\
+1x^3 - 5x^2 + x + 9 & \quad \text{principal part of the quotient} \\
\text{that has Concavity:} \\
+1x + \ldots
\end{align*} \]

\[ \begin{align*}
\text{i. Approximate} \\
+1x^2 - 4x + 3 & \quad \text{dividing into} \\
+1x^3 - 5x^2 + x + 9 & \quad \text{principal part of the quotient} \\
\text{that has Concavity:} \\
+1x + \ldots
\end{align*} \]

\[ \begin{align*}
\text{i. Approximate} \\
x - 1 - 6x^{-1} & \quad \text{since it is the term responsible for Concavity.}
\end{align*} \]

The more usual way to write all this is:
CHAPTER 16. RATIONAL FUNCTIONS: LOCAL ANALYSIS NEAR $\infty$

$$x \bigg|_{x \to \infty} \xrightarrow{\text{BATH}} BATH(x) \bigg|_{x \to \infty} = \frac{+x^3 - 5x^2 + x + 9}{+x^2 - 4x + 3}$$

and then we divide (in the Latin manner):

$$\begin{array}{rrrrr}
+1 & -1 & -6x^{-1} & +[...] \\
+1 & -5x^2 & +x & +9 \\
+1 & -4x^2 & +3x \\
0x^4 & -x^2 & -2x & +9 & \\
0x^4 & -x^2 & +4x & -3 & +12
\end{array}$$

Whichever way we write it, $\text{Princ. PART BATH} \bigg|_{x \to \infty} = +x - 1 - 6x^{-1}$ and its third term, $-6x^{-1}$, gives

Concavity-sign $BATH \bigg|_{x \to \infty} = (\cap, \cup)$

16.5 Local Graph Near $\infty$

In order to get the local graph near $\infty$, we need a local input-output rule that gives us the concavity-sign and therefore the slope-sign and the height-sign.

So, the route we must take in order to get the local graph near $\infty$ is the route that will get us the concavity-sign near $\infty$.

**EXAMPLE 6.** Given the rational function $SOUTH$ whose global input-output rule is

$$x \xrightarrow{\text{SOUTH}} SOUTH(x) = \frac{-3x^2 - 5x + 6}{12x^4 - 6x^3 + 8x^2 + 6x - 9}$$

find its local graph near $\infty$.

i. We get the local input-output rule near $\infty$ as in Example 1. We have:

$$x \bigg|_{x \to \infty} \xrightarrow{\text{SOUTH}} SOUTH(x) \bigg|_{x \to \infty} = \frac{-3x^2 - 5x + 6}{12x^4 - 6x^3 + 8x^2 + 6x - 9}$$

We approximate separately the numerator and the denominator:

$$= \frac{-3x^2 + [\ldots]}{+12x^5 + [\ldots]}$$
We divide the approximations:

\[ \frac{-3}{+12} x^{2-5} + [...] = -\frac{1}{4} x^{-3} + [...] \]

ii. Since the degree of the power function

\[ x \xrightarrow{\text{POWER}} \text{POWER}(x) = -\frac{1}{4} x^{-3} \]

is \( < 0 \), the power function \( \text{POWER} \) is regular and has both concavity and slope. So, the local graph of the power function \( \text{POWER} \) near \( \infty \) will be approximately the graph near \( \infty \) of the rational function \( \text{SOUTH} \).

The local graph near \( \infty \) of the rational function \( \text{SOUTH} \) is therefore:

**Example 7.** Given the rational function \( \text{DOUGH} \) whose global input-output rule is

\[ x \xrightarrow{\text{DOUGH}} \text{DOUGH}(x) = \frac{+12 x^4 - 6 x^3 + 8 x^2 + 6 x - 9}{-3 x^2 - 5 x + 6} \]

find its local graph near \( \infty \).

i. We get the local input-output rule near \( \infty \).

We have:

\[ x \bigg|_{x \nearrow \infty} \xrightarrow{\text{DOUGH}} \text{DOUGH}(x) \bigg|_{x \nearrow \infty} = \frac{+12 x^5 - 6 x^3 + 8 x^2 + 6 x - 9}{-3 x^2 - 5 x + 6} \bigg|_{x \nearrow \infty} \]

We approximate separately the numerator and the denominator:

\[ = \frac{+12 x^5 + [...]}{-3 x^4 + [...]} \]

We divide the approximations:

\[ = \frac{-12}{-3} x^{5-2} + [...] \]

\[ = -4 x^3 + [...] \]
ii. Since the degree of the power function

\[ x \xrightarrow{\text{POWER}} \text{POWER}(x) = -4x^3 \]

is > 1, the power function \( \text{POWER} \) is regular and has both concavity and slope. So, the local graph of the power function \( \text{POWER} \) near \( \infty \) will be approximately the graph near \( \infty \) of the rational function \( \text{DOUGH} \).

The local graph near \( \infty \) of the rational function \( \text{DOUGH} \) is therefore:

**Example 8.** Given the rational function \( \text{BATH} \) specified by the global input-output rule

\[ x \xrightarrow{\text{BATH}} \text{BATH}(x) = \frac{+x^3 + x^2 - 5x + 6}{+x^2 - 4x + +3} \]

as in **Example 1**, find the local graph near \( \infty \).

i. We get the local input-output rule near \( \infty \) that gives all three features as we did in **Example 1**:

\[ x|_{x \rightarrow \infty} \xrightarrow{\text{BATH}} \text{BATH}(x)|_{x \rightarrow \infty} = +x + 5 + 27x^{-1} + [...] \]

ii. So the local graph near \( \infty \) of the function \( \text{BATH} \) is