Chapter 16

Rational Functions: Local Analysis Near ∞

To do local analysis we work in a neighborhood of some given input and thus count inputs from the given input since it is the center of the neighborhood. When the given input is ∞, counting from ∞ means setting $x \leftarrow$ *large* and computing with powers of *large* in descending order of sizes.

Recall that the *principal term* near ∞ of a given polynomial function $POLY$ is simply its highest power term which is therefore easy to **extract** from the global input-output rule. The approximate input-output rule near ∞ of $POLY$ is then of the form

$$x|_{x \text{ near } \infty} \xrightarrow{POLY} POLY(x)|_{x \text{ near } \infty} = Highest\,Term\;POLY + [...]$$

However, the complication here is that to get the principal part near ∞ of a rational function we must approximate the two polynomial and divide—or the other way round—and the result need not be a polynomial but can also be a negative-exponent power function and the main issue will be whether to do the approximation before or after the division.

16.1 Local Input-Output Rule Near ∞

Given a rational function RAT, we look for the function whose input-output rule will be simpler than the input-output rule of RAT but whose local graph near ∞ will be qualitatively the same as the local graph near ∞ of RAT.

More precisely, given a rational function RAT specified by the global input-output rule

$$x \xrightarrow{RAT} RAT(x) = \frac{POLY_{Num}(x)}{POLY_{Den}(x)}$$

what we will want then is an *approximation* for the output of the local input-output rule near ∞

$$x|_{x \text{ near } \infty} \xrightarrow{RAT} RAT(x)|_{x \text{ near } \infty} = \left.\frac{POLY_{Num}(x)}{POLY_{Den}(x)}\right|_{x \text{ near } \infty}$$

from which to *extract* whatever controls the wanted feature.

1. Since the center of the neighborhood is ∞, we *localize* both

- $POLY_{Num}(x)$

and

- $POLY_{Den}(x)$

by writing them in *descending* order of exponents.

$$\frac{POLY_{Num}(x)}{POLY_{Den}(x)} \quad \begin{matrix} \xrightarrow{\text{Localize near } \infty} \\ \xrightarrow{\text{Localize near } \infty} \end{matrix} \quad \frac{POLY_{Num}(x)|_{x \text{ near } \infty}}{POLY_{Den}(x)|_{x \text{ near } \infty}}$$

2. Depending on the circumstances, we will take one of the following two routes to *extract* what controls the wanted feature:

■ The *short route* to $Princ.TERM\ RAT(x)\ |_{x \text{ near } \infty}$, that is:

i. We approximate both $POLY_{Num}(x)|_{x \text{ near } \infty}$ and $POLY_{Den}(x)|_{x \text{ near } \infty}$ to their *principal term*—that is to just their *highest size term*—which, since x is near ∞, is their *highest exponent term*:

$$\frac{POLY_{Num}(x)}{POLY_{Den}(x)} \quad \begin{matrix} \xrightarrow{\text{Localize near } \infty} \\ \xrightarrow{\text{Localize near } \infty} \end{matrix} \quad \frac{POLY_{Num}(x)|_{x \text{ near } \infty}}{POLY_{Den}(x)|_{x \text{ near } \infty}} \quad \begin{matrix} \xrightarrow{\text{i. Approximate}} \\ \xrightarrow{\text{i. Approximate}} \end{matrix} \quad \frac{Princ.TERM_{Num}(x)\,|_{x \text{ near } \infty} + [...]}{Princ.TERM_{Den}(x)|_{x \text{ near } \infty} + [...]}$$

ii. In order to divide $Princ.TERM_{Num(x)}\Big|_{x \text{ near } \infty}$, that is the principal term near ∞ of the *numerator* of RAT by $Princ.TERM_{Den(x)}\Big|_{x \text{ near } \infty}$, that is the principal term near ∞ of the *denominator* of RAT we use monomial division

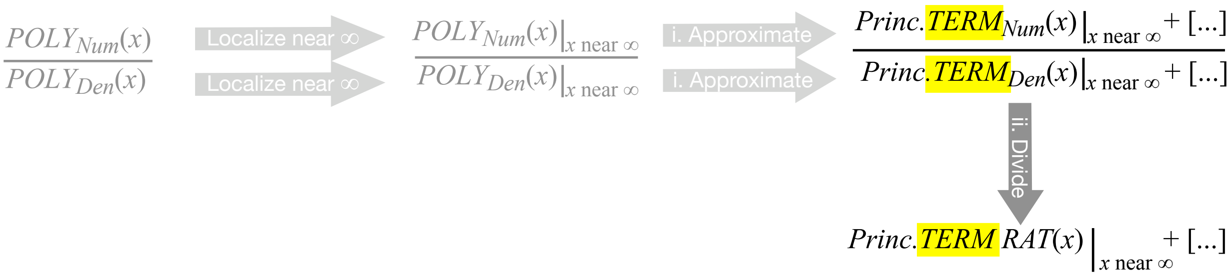
$$\boxed{\frac{ax^{+m}}{bx^{+n}} = \frac{a}{b}x^{+m \ominus +n}}$$ where $+m \ominus +n$ can turn out positive, negative or 0

$$Princ.TERM\ RAT(x)\Big|_{x \text{ near } \infty} = \frac{Princ.TERM_{Num}(x)\Big|_{x \text{ near } \infty}}{Princ.TERM_{Den}(x)\Big|_{x \text{ near } \infty}}$$

$$= \frac{\text{coef. } Princ.TERM_{Num}(x)\Big|_{x \text{ near } \infty}}{\text{coef. } Princ.TERM_{Den}(x)\Big|_{x \text{ near } \infty}} \cdot x^{\text{UppDeg}.POLY_{Num}(x)-\text{UppDeg}.POLY_{Den}(x)}$$

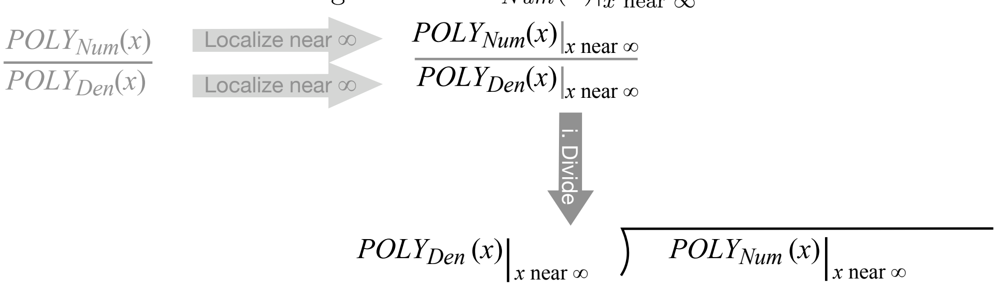
$$= \frac{\text{coef. } Princ.TERM_{Num}(x)\Big|_{x \text{ near } \infty}}{\text{coef. } Princ.TERM_{Den}(x)\Big|_{x \text{ near } \infty}} \cdot x^{\text{RatDeg}.RAT(x)}$$

The resulting monomial is $Princ.TERM\,RAT(x)\Big|_{x \text{ near } \infty}$, that is the *principal term* of the rational function RAT near ∞:

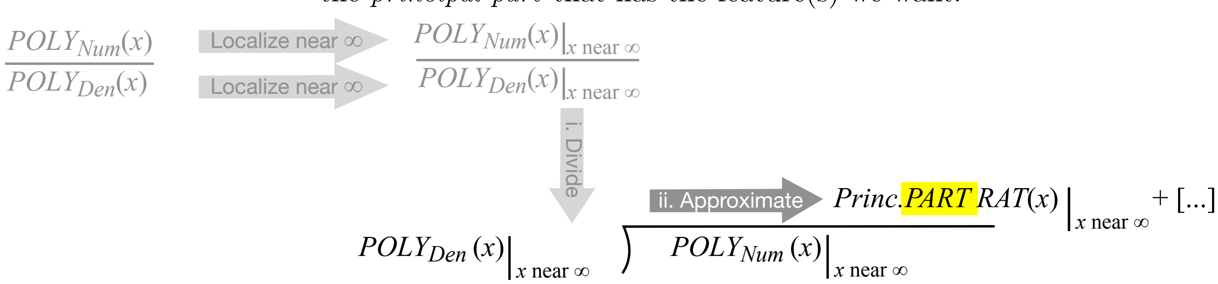


■ The *long route* to $Princ.PART\;RAT(x)\,|_{x \text{ near } \infty}$:

i. In order to divide $POLY_{Num}(x)|_{x \text{ near } \infty}$ *by* $POLY_{Den}(x)|_{x \text{ near } \infty}$, we set up the division as a *long division*, that is $POLY_{Den}(x)|_{x \text{ near } \infty}$ dividing *into* $POLY_{Num}(x)|_{x \text{ near } \infty}$:



ii. We approximate by stopping the long division as soon as we have the *principal part* that has the feature(s) we want:



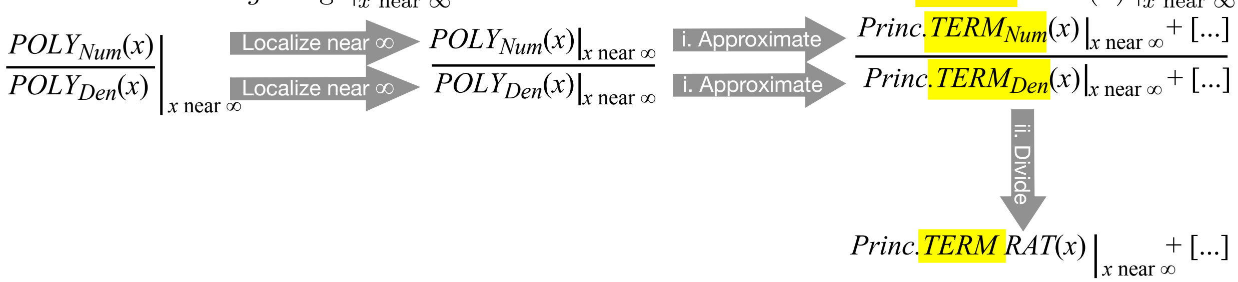
3. Which route we will take in each particular case will depend both on the *wanted feature(s)* near ∞ and on the *rational degree* of RAT and so we will now look separately at how we get $Height$-sign$|_{x \text{ near } \infty}$, $Slope$-sign$|_{x \text{ near } \infty}$ and $Concavity$-sign$|_{x \text{ near } \infty}$

LOCAL ANALYSIS NEAR ∞

When the wanted features are to be found near ∞, the *rational degree* of the rational function tells us up front whether or not the *short route* will allow us to extract the term that controls the wanted feature.

16.2 Height-sign Near ∞

No matter what the *rational degree* of the given rational function RAT, $Princ.TERM\ RAT(x)\ |_{x \text{ near } \infty}$ will give us $Height$-sign$|_{x \text{ near } \infty}$ because, no matter what its exponent, *any* power function has $Height$-sign$|_{x \text{ near } \infty}$. So, no matter what the *rational degree* of RAT, to *extract* the term responsible for $Height$-sign$|_{x \text{ near } \infty}$ we can take the *short route* to $Princ.TERM\ RAT(x)\ |_{x \text{ near } \infty}$:

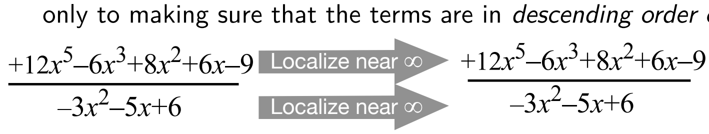


EXAMPLE 1. Given the rational function $DOUGH$ specified by the global input-output rule

$$x \xrightarrow{DOUGH} DOUGH(x) = \frac{+12x^5 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6}$$

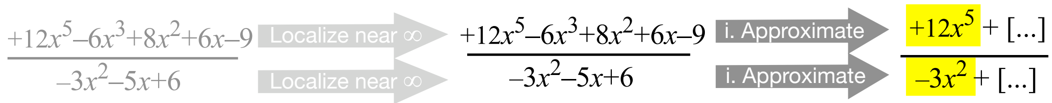
find $Height$-sign $DOUGH|_{x \text{ near } \infty}$.

a. We localize both the numerator and the denominator near ∞—which amounts only to making sure that the terms are in *descending order of exponents*.



b. Inasmuch as $Princ.TERM\ DOUGH(x)\ |_{x \text{ near } \infty}$ has *Height* no matter what the degree, in order to *extract* the term that controls $Height$-sign$|_{x \text{ near } \infty}$ we take the short route to $Princ.TERM\ DOUGH(x)\ |_{x \text{ near } \infty}$:

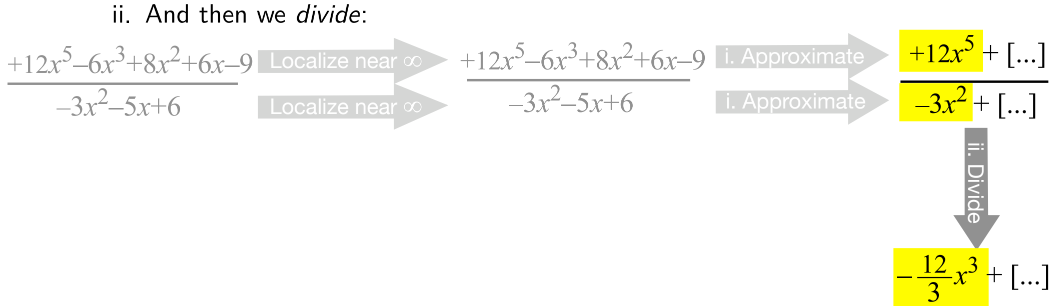
i. We *approximate*



that is we approximate

- the numerator $+12x^5 - 6x^3 + 8x^2 + 6x - 9$ to its *principal term*, $-12x^5$
- the denominator $-3x^2 - 5x + 6$ to its *principal term*, $-3x^2$

ii. And then we *divide*:



where

$$\frac{+12x^5}{-3x^2} = \frac{+12 \cdot x \cdot x \cdot x \cdot x \cdot x}{-3 \cdot x \cdot x}$$
$$= -\frac{12}{3}x^{5-2}$$

The more usual way to write all this is something as follows:

$$x\big|_{x \text{ near } \infty} \xrightarrow{DOUGH} DOUGH(x)\big|_{x \text{ near } \infty} = \left.\frac{+12x^5 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6}\right|_{x \text{ near } \infty}$$
$$= \frac{+12x^5 - 6x^3 + 8x^2 + 6x - 9\big|_{x \text{ near } \infty}}{-3x^2 - 5x + 6\big|_{x \text{ near } \infty}}$$
$$= \frac{+12x^5 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6}$$
$$= \frac{+12x^5 + [...]}{-3x^2 + [...]}$$
$$= -\frac{12}{3}x^{5-2} + [...]$$

Whatever we write, the *principat term* of $DOUGH$ near ∞ is $-\frac{12}{3}x^3$ and it gives

$$Height\text{-sign } DOUGH\big|_{x \text{ near } \infty} = (-, +)$$

Example 2. Given the function PAC specified by the global input-output rule

$$x \xrightarrow{PAC} PAC(x) = \frac{-12x^3 + 7x + 4}{+4x^5 - 6x^4 - 17x^2 - 2x + 10}$$

find Height-sign $PAC|_{x \text{ near } \infty}$.

Inasmuch as $Princ.TERM\ PAC(x)\ |_{x \text{ near } \infty}$ has *Height* no matter what the degree, in order to *extract* the term that controls $Height$-sign$|_{x \text{ near } \infty}$ we take the short route to $Princ.TERM\ DOUGH(x)\ |_{x \text{ near } \infty}$:

$$
\begin{aligned}
x|_{x \text{ near } \infty} \xrightarrow{\ PAC\ } PAC(x)|_{x \text{ near } \infty} &= \left.\frac{-12x^3+7x+4}{+4x^5-6x^4-17x^2-2x+10}\right|_{x \text{ near } \infty} \\
&= \frac{-12x^3+7x+4|_{x \text{ near } \infty}}{+4x^5-6x^4-17x^2-2x+10|_{x \text{ near } \infty}} \\
&= \frac{-12x^{+3}+[...]}{+4x^{+5}+[...]} \\
&= \frac{-12}{+4}x^{+3\ominus+5}+[...] \\
&= -3x^{-2}+[...]
\end{aligned}
$$

and we get that

$$\text{Height-sign } PAC|_{x \text{ near } \infty} = (-,-)$$

16.3 Slope-sign Near ∞

In the case of $Slope$-sign $RAT|_{x \text{ near } \infty}$, there are two cases depending on the *rational degree* of the given rational function:

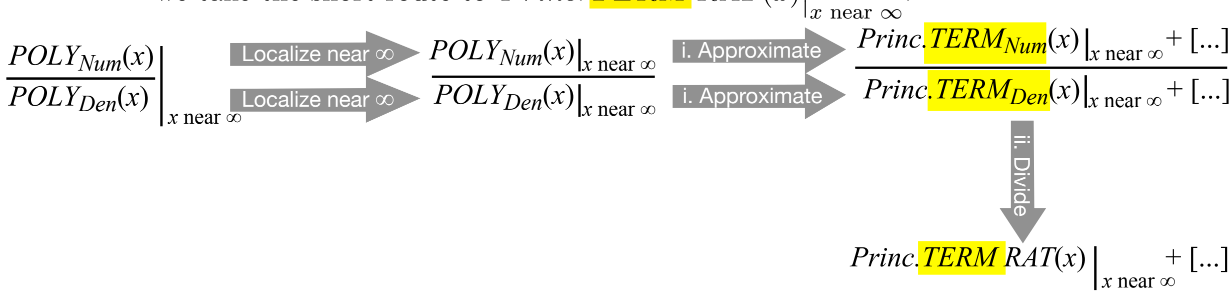
■ If the rational functionRAT is either:

– A *regular* rational function, that is of rational degree > 1 or < 0

or

– An *exceptional* rational function of rational degree $= 1$,

that is *not* an exceptional rational function of rational degree $= 0$, then $Princ.\ TERM\ RAT(x)\Big|_{x \text{ near } \infty}$ will be a *power function* that will have *Slope* near ∞ and so in order to *extract* the term that controls $Slope$-sign$|_{x \text{ near } \infty}$ we take the short route to $Princ.\ TERM\ RAT(x)\Big|_{x \text{ near } \infty}$:



EXAMPLE 3. Given the rational function $SOUTH$ specified by the global input-output rule

$$x \xrightarrow{SOUTH} SOUTH(x) = \frac{-3x^2 - 5x + 6}{+12x^4 - 6x^3 + 8x^2 + 6x - 9}$$

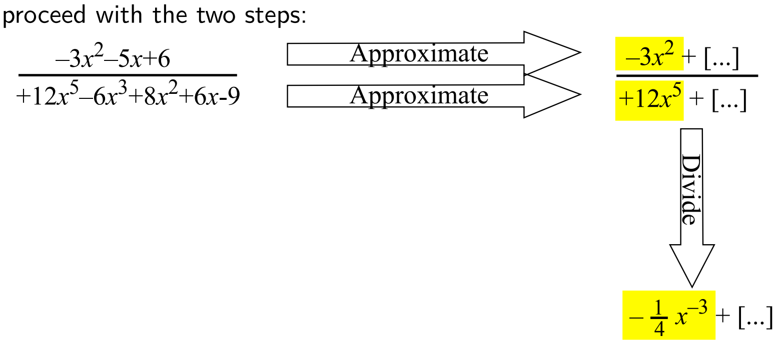
find Slope-sign of $SOUTH$ near ∞

i. We get the local graph near ∞ of $SOUTH$

a. We have

$$x|_{x \text{ near } \infty} \xrightarrow{SOUTH} SOUTH(x)|_{x \text{ near } \infty} = \left.\frac{-3x^2 - 5x + 6}{+12x^5 - 6x^3 + 8x^2 + 6x - 9}\right|_{x \text{ near } \infty}$$
$$= \frac{-3x^2 - 5x + 6|_{x \text{ near } \infty}}{+12x^5 - 6x^3 + 8x^2 + 6x - 9|_{x \text{ near } \infty}}$$

We now proceed with the two steps:



b. The more usual presentation is:

$$x|_{x \text{ near } \infty} \xrightarrow{SOUTH} SOUTH(x)|_{x \text{ near } \infty} = \left.\frac{-3x^2 - 5x + 6}{+12x^5 - 6x^3 + 8x^2 + 6x - 9}\right|_{x \text{ near } \infty}$$
$$= \frac{-3x^2 - 5x + 6|_{x \text{ near } \infty}}{+12x^5 - 6x^3 + 8x^2 + 6x - 9|_{x \text{ near } \infty}}$$

We *approximate* $-3x^2 - 5x + 6|_{x \text{ near } \infty}$ and $+12x^5 - 6x^3 + 8x^2 + 6x - 9|_{x \text{ near } \infty}$

$$= \frac{-3x^2 + [...]}{+12x^5 + [...]}$$

and then we *divide*:

$$= \frac{-3}{+12}x^{2-5} + [...]$$
$$= -\frac{1}{4}x^{-3} + [...]$$

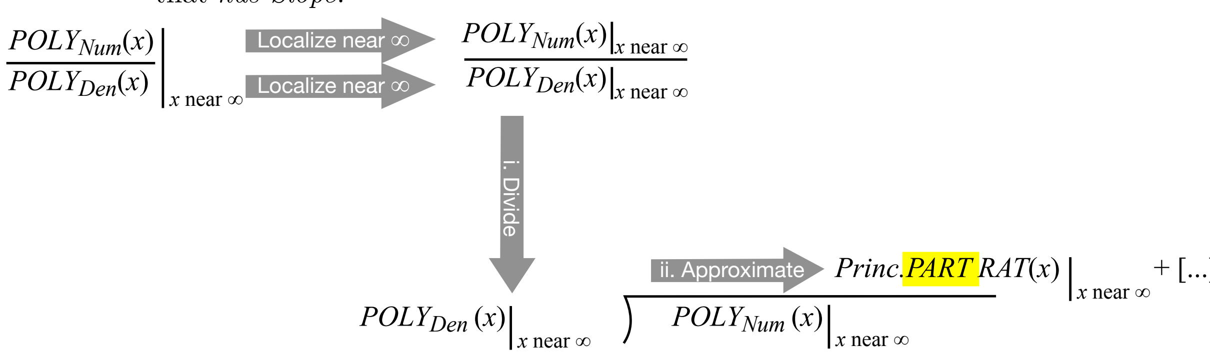
c. Since the degree of the power function

$$x \xrightarrow{POWER} POWER(x) = -\frac{1}{4}x^{-3}$$

which approximates $SOUTH$ near ∞ is < 0, the power function $POWER$ has all three features, *concavity*, *slope* and *height*. (This was of course to be expected from the fact that the *rational degree* of $SOUTH$ is < 0.)
ii. We get

$$\text{Slope-sign of } SOUTH \text{ near } \infty = (\diagup, \diagdown)$$

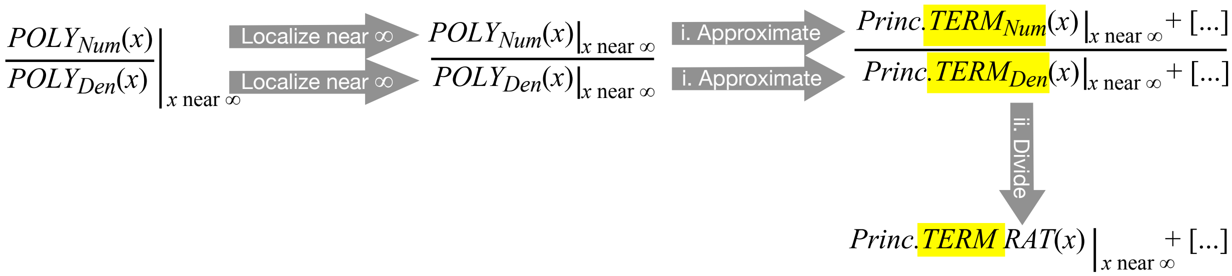
■ If the rational function RAT *is* an *exceptional rational function* whose rational degree $= 0$, then $Princ.TERM\ RAT(x)\ |_{x \text{ near } \infty}$ will be an *exceptional power function* with exponent $= 0$ and $Princ.TERM\ RAT(x)\ |_{x \text{ near } \infty}$ will *not* have *Slope* and so in order to *extract* the term that controls *Slope*-sign$|_{x \text{ near } \infty}$ we will have to take the long route to a $Princ.PART\ RAT(x)\ |_{x \text{ near } \infty}$ that *has Slope*:



16.4 Concavity-sign Near ∞

In the case of *Concavity*-sign $RAT|_{x \text{ near } \infty}$, there are *two* cases depending on the rational degree of the given rational function.

■ If the rational function RAT is a *regular* rational function, that is if the rational degree of RAT is either > 1 or < 0, then $Princ.\ TERM\ RAT(x)\ |_{x \text{ near } \infty}$ will be a *regular power function*, that is a power function whose exponent is either > 1 or < 0 and then, in either case, $Princ.\ TERM\ RAT(x)\ |_{x \text{ near } \infty}$ will have *Concavity* and so in order to *extract* the term that controls *Concavity*-sign$|_{x \text{ near } \infty}$ we take the short route to $Princ.TERM_{Den}(x)\ |_{x \text{ near } \infty}$:



Example 4. Given the rational function $SOUTH$ specified by the global input-output rule

$$x \xrightarrow{SOUTH} SOUTH(x) = \frac{-3x^2 - 5x + 6}{+12x^4 - 6x^3 + 8x^2 + 6x - 9}$$

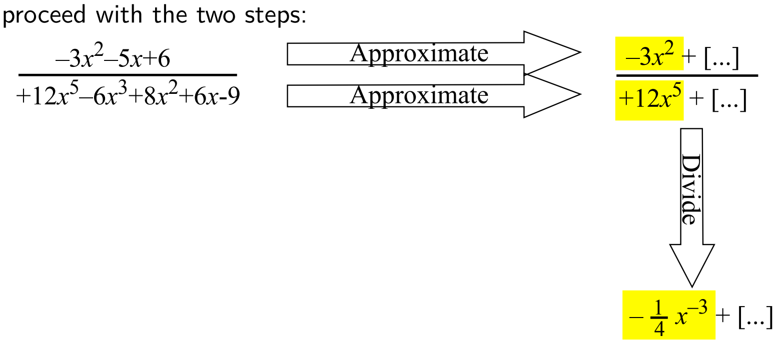
find Concavity-sign of $SOUTH$ near ∞

i. We get the local graph near ∞ of $SOUTH$

a. We have

$$x|_{x \text{ near } \infty} \xrightarrow{SOUTH} SOUTH(x)|_{x \text{ near } \infty} = \left.\frac{-3x^2 - 5x + 6}{+12x^5 - 6x^3 + 8x^2 + 6x - 9}\right|_{x \text{ near } \infty}$$
$$= \frac{-3x^2 - 5x + 6|_{x \text{ near } \infty}}{+12x^5 - 6x^3 + 8x^2 + 6x - 9|_{x \text{ near } \infty}}$$

We now proceed with the two steps:



b. The more usual presentation is:

$$x|_{x \text{ near } \infty} \xrightarrow{SOUTH} SOUTH(x)|_{x \text{ near } \infty} = \left.\frac{-3x^2 - 5x + 6}{+12x^5 - 6x^3 + 8x^2 + 6x - 9}\right|_{x \text{ near } \infty}$$
$$= \frac{-3x^2 - 5x + 6|_{x \text{ near } \infty}}{+12x^5 - 6x^3 + 8x^2 + 6x - 9|_{x \text{ near } \infty}}$$

We *approximate* $-3x^2 - 5x + 6\big|_{x \text{ near } \infty}$ and $+12x^5 - 6x^3 + 8x^2 + 6x - 9\big|_{x \text{ near } \infty}$

$$= \frac{-3x^2 + [...]}{+12x^5 + [...]}$$

and then we *divide*:

$$= \frac{-3}{+12}x^{2-5} + [...]$$
$$= -\frac{1}{4}x^{-3} + [...]$$

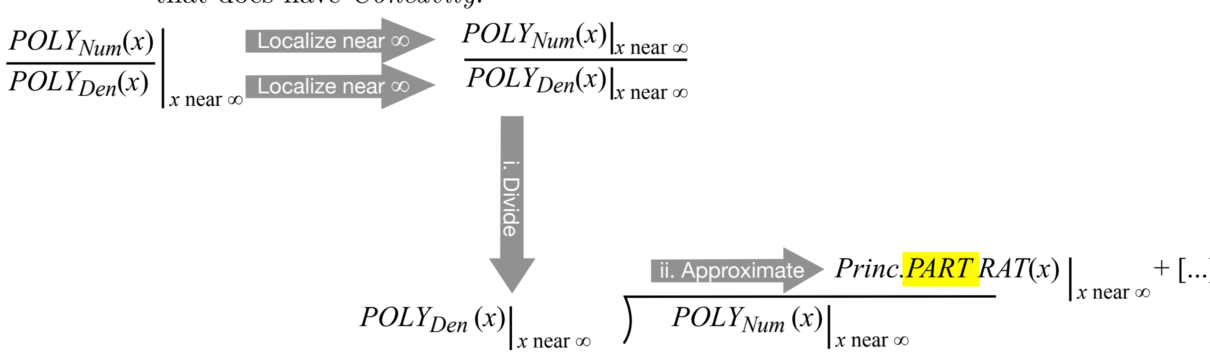
c. Since the degree of the power function

$$x \xrightarrow{POWER} POWER(x) = -\frac{1}{4}x^{-3}$$

which approximates $SOUTH$ near ∞ is < 0, the power function $POWER$ has all three features, *concavity*, *slope* and *height*. (This was of course to be expected from the fact that the *rational degree* of $SOUTH$ is < 0.)
ii. We get

Concavity-sign of $SOUTH$ near $\infty = (\cap, \cap)$

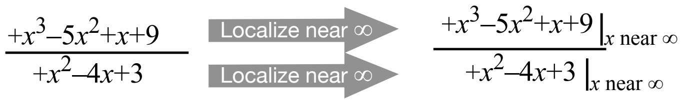
■ If the rational function RAT is an *exceptional* rational function that is if the rational degree of RAT is either $= 1$ or $= 0$ then $Princ.TERM\ RAT(x)\ |_{x \text{ near } \infty}$ will be an *exceptional power function* with exponent either $= 1$ or $= 0$ (**Chapter 7**) and in both cases $Princ.TERM\ RAT(x)\ |_{x \text{ near } \infty}$ will *not* have *Concavity* and in order *extract* the term that controls $Concavity\text{-sign}|_{x \text{ near } \infty}$ we will have to take the long route to a $Princ.PART\ RAT(x)\ |_{x \text{ near } \infty}$ that does have *Concavity*.



EXAMPLE 5. Given the rational function $BATH$ specified by the global input-output rule

$$x \xrightarrow{BATH} BATH(x) = \frac{+x^3 - 5x^2 + x + 6}{+x^2 - 4x + 3}$$

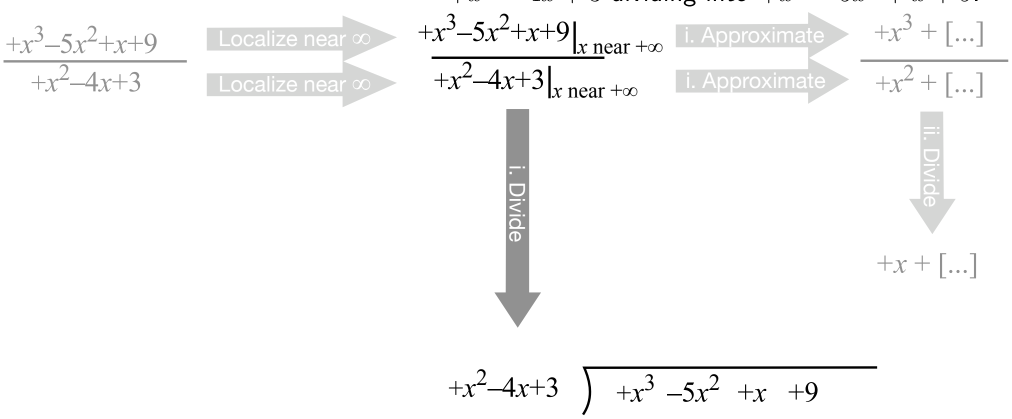
find $Concavity\text{-sign } BATH|_{x \text{ near } \infty}$.
a. The *localization step* is to *localize* both the numerator and the denominator near ∞—which amounts only to making sure that the terms are in *descending order of exponents*.



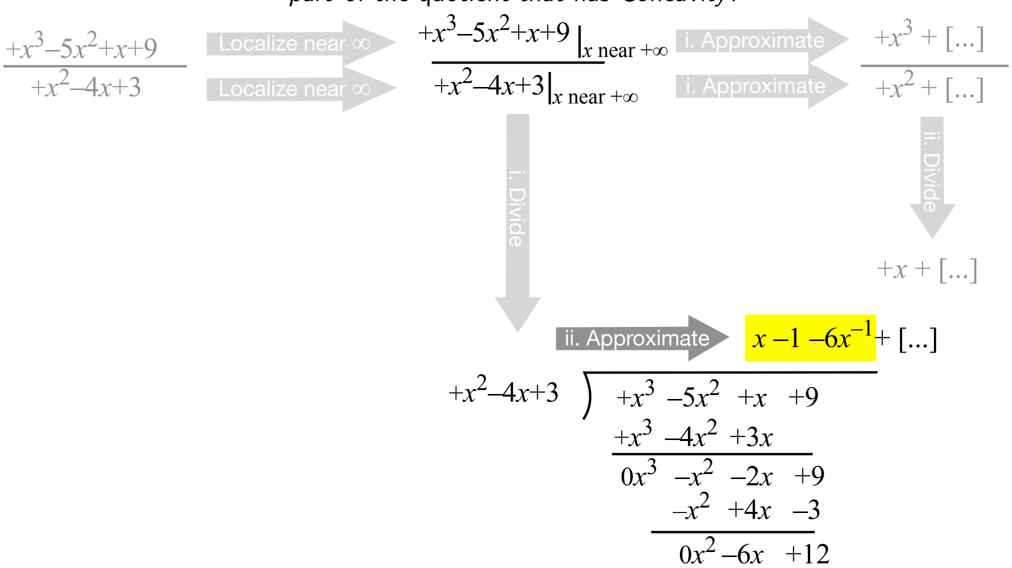
b. Since $Princ.\ TERM\ BATH(x)\ |_{x \text{ near } \infty}$ has no *Concavity*, the *extraction step* to get $Concavity$-sign $BATH|_{x \text{ near } \infty}$ must take the long route to a $Princ.\ PART\ BATH(x)\ |_{x \text{ near } \infty}$ that has *Concavity*:

i. We set up the division as a *long division*:

$+x^2 - 4x + 3$ dividing *into* $+x^3 - 5x^2 + x + 9$.



ii. We *approximate* by stopping the long division as soon as we have the *principal part* of the quotient that has *Concavity*:



that is we stop with $-6x^{-1}$ since it is the term responsible for *Concavity*.

The more usual way to write all this is:

$$x|_{x \text{ near } \infty} \xrightarrow{BATH} BATH(x)|_{x \text{ near } \infty} = \left.\frac{+x^3 - 5x^2 + x + 9}{+x^2 - 4x + 3}\right|_{x \text{ near } \infty}$$
$$= \frac{+x^3 - 5x^2 + x + 9|_{x \text{ near } \infty}}{+x^2 - 4x + 3|_{x \text{ near } \infty}}$$
$$= \frac{+x^3 - 5x^2 + x + 9}{+x^2 - 4x + 3}$$

and then we *divide* (in the *latin* manner):

$$\begin{array}{r|llll}
 & +x & -1 & -6x^{-1} & +[...] \\
\hline
+x^2 \ -4x \ +3 & +x^3 & -5x^2 & +x & +9 \\
 & +x^3 & -4x^2 & +3x & \\
\hline
 & 0x^3 & -x^2 & -2x & +9 \\
 & & -x^2 & +4x & -3 \\
\hline
 & & 0x^2 & -6x & +12
\end{array}$$

Whichever way we write it, $Princ.\ PART\ BATH(x)\ |_{x \text{ near } \infty} = +x - 1 - 6x^{-1}$ and its third term, $-6x^{-1}$, gives

$$Concavity\text{-sign } BATH|_{x \text{ near } \infty} = (\cap, \cup)$$

16.5 Local Graph Near ∞

In order to get the local graph near ∞, we need a local input-output rule that gives us the *concavity*-signÑand therefore the *slope*-sign and the *height*-sign.

So, the route we must take in order to get the local graph near ∞ is the route that will get us the concavity-sign near ∞.

EXAMPLE 6. Given the rational function $SOUTH$ whose global input-output rule is

$$x \xrightarrow{SOUTH} SOUTH(x) = \frac{-3x^2 - 5x + 6}{+12x^4 - 6x^3 + 8x^2 + 6x - 9}$$

find its local graph near ∞.

i. We get the *local input-output rule* near ∞ as in EXAMPLE 1.

We have:

$$x|_{x \text{ near } \infty} \xrightarrow{SOUTH} SOUTH(x)|_{x \text{ near } \infty} = \left.\frac{-3x^2 - 5x + 6}{+12x^5 - 6x^3 + 8x^2 + 6x - 9}\right|_{x \text{ near } \infty}$$
$$= \frac{-3x^2 - 5x + 6|_{x \text{ near } \infty}}{+12x^5 - 6x^3 + 8x^2 + 6x - 9|_{x \text{ near } \infty}}$$

We *approximate* separately the *numerator* and the *denominator*:

$$= \frac{-3x^2 + [...]}{+12x^5 + [...]}$$

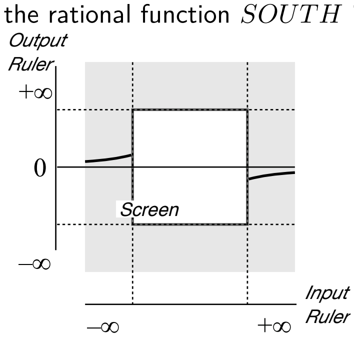
We *divide* the approximations:

$$= \frac{-3}{+12} x^{2-5} + [...]$$
$$= -\frac{1}{4} x^{-3} + [...]$$

ii. Since the degree of the power function

$$x \xrightarrow{POWER} POWER(x) = -\frac{1}{4} x^{-3}$$

is < 0, the power function $POWER$ is *regular* and has both *concavity* and *slope*. So, the local graph of the power function $POWER$ near ∞ will be approximately the graph near ∞ of the rational function $SOUTH$.
The local graph near ∞ of the rational function $SOUTH$ is therefore:



EXAMPLE 7. Given the rational function $DOUGH$ whose global input-output rule is

$$x \xrightarrow{DOUGH} DOUGH(x) = \frac{+12x^4 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6}$$

find its local graph near ∞.
i. We get the *local input-output rule* near ∞.
We have:

$$x|_{x \text{ near } \infty} \xrightarrow{DOUGH} DOUGH(x)|_{x \text{ near } \infty} = \left. \frac{+12x^5 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6} \right|_{x \text{ near } \infty}$$
$$= \frac{+12x^5 - 6x^3 + 8x^2 + 6x - 9|_{x \text{ near } \infty}}{-3x^2 - 5x + 6|_{x \text{ near } \infty}}$$

We *approximate* separately the *numerator* and the *denominator*:

$$= \frac{+12x^5 + [...]}{-3x^4 + [...]}$$

We *divide* the approximations:

$$= -\frac{+12}{-3} x^{5-2} + [...]$$
$$= -4x^{+3} + [...]$$

ii. Since the degree of the power function

$$x \xrightarrow{POWER} POWER(x) = -4x^{+3}$$

is > 1, the power function $POWER$ is *regular* and has both *concavity* and *slope*. So, the local graph of the power function $POWER$ near ∞ will be approximately the graph near ∞ of the rational function $DOUGH$.
The local graph near ∞ of the rational function $DOUGH$ is therefore:

Example 8. Given the rational function $BATH$ specified by the global input-output rule

$$x \xrightarrow{BATH} BATH(x) = \frac{+x^3 + x^2 - 5x + 6}{+x^2 - 4x + +3}$$

as in Example 1, find the local graph near ∞.

i. We get the *local input-output rule* near ∞ that gives all three features as we did in Example 1:

$$x|_{x \text{ near } \infty} \xrightarrow{BATH} BATH(x)|_{x \text{ near } \infty} = +x + 5 + 27x^{-1} + [...]$$

ii. So the local graph near ∞ of the function $BATH$ is