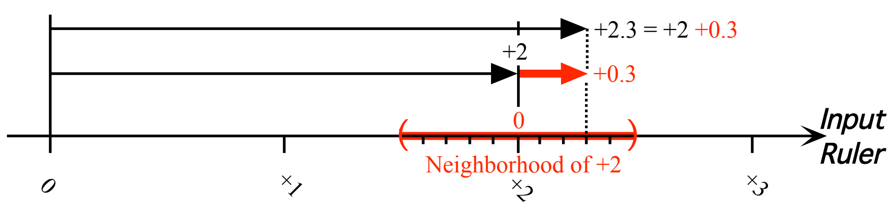
Chapter 17

Rational Functions: Local Analysis Near x_0

Doing local analysis means working in a neighborhood of some given input and thus counting inputs from the given input since it is the *center* of the neighborhood. When the given input is x_0, we *localize* at x_0, that is we set $x = x_0 + h$ where h is *small* and we compute with powers of h in descending order of sizes.

EXAMPLE 1. Given the input $+2$, then the location of the number $+2.3$ relative to $+2$ is $+0.3$:



Recall that the *principal part* near x_0 of a given polynomial function $POLY$ is the local quadratic part

$$x|_{x \text{ near } x_0} \xrightarrow{POLY} POLY(x)|_{x \text{ near } x_0} = \Big[\quad\Big] + \Big[\quad\Big]h + \Big[\quad\Big]h^2 + [...]$$

However, the complication here is that to get the principal part near x_0 of a rational function we must approximate the two polynomial and divide—or the other way round—and the result need not be a polynomial but can also be a negative-exponent power function and the main issue will be whether to do the approximation before or after the division.

17.1 Local Input-Output Rule Near x_0

Given a rational function RAT, we look for the function whose input-output rule will be simpler than the input-output rule of RAT but whose local graph near x_0 will be qualitatively the same as the local graph near x_0 of RAT.

More precisely, given a rational function RAT specified by the global input-output rule

$$x \xrightarrow{RAT} RAT(x) = \frac{POLY_{Num}(x)}{POLY_{Den}(x)}$$

what we will want then is an *approximation* for the output of the local input-output rule near x_0

$$x\big|_{x \text{ near } x_0} \xrightarrow{RAT} RAT(x)\big|_{x \text{ near } x_0} = \frac{POLY_{Num}(x)}{POLY_{Den}(x)}\bigg|_{x \text{ near } x_0}$$

from which to *extract* whatever controls the wanted feature.

1. Since the center of the neighborhood is x_0, we *localize* both

- $POLY_{Num}(x)$

and

- $POLY_{Den}(x)$

by letting $x \leftarrow x_0 + h$ and writing the terms in *ascending* order of exponents.

$$\frac{POLY_{Num}(x)}{POLY_{Den}(x)} \quad \begin{matrix} \xrightarrow{\text{Localize near } x_0} \\ \xrightarrow{\text{Localize near } x_0} \end{matrix} \quad \frac{POLY_{Num}(x)\big|_{x \text{ near } x_0}}{POLY_{Den}(x)\big|_{x \text{ near } x_0}}$$

2. Depending on the circumstances, we will take one of the following two routes to *extract* what controls the wanted feature:

■ The *short route* to $Princ.TERM\ RAT(x)\ \big|_{x \text{ near } x_0}$, that is:

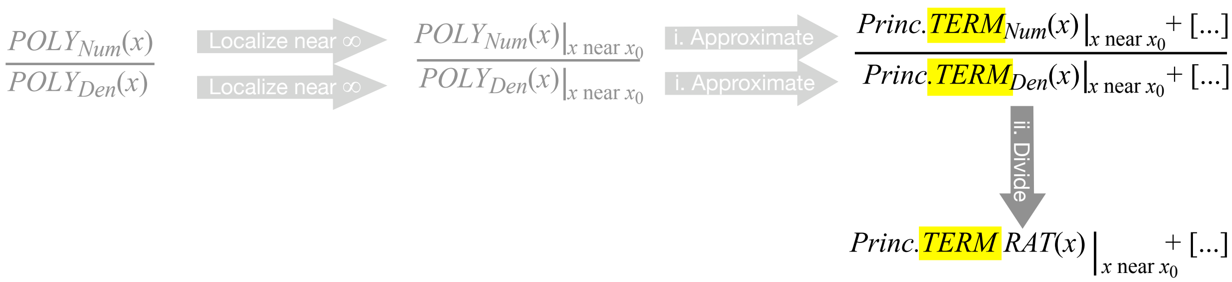
i. We approximate both $POLY_{Num}(x)\big|_{x \text{ near } x_0}$ and $POLY_{Den}(x)\big|_{x \text{ near } x_0}$ to their *principal term*—that is to just their *lowest size term*—which, since x is near ∞, is their *lowest exponent term*:

$$\frac{POLY_{Num}(x)}{POLY_{Den}(x)} \quad \begin{matrix} \xrightarrow{\text{Localize near } \infty} \\ \xrightarrow{\text{Localize near } \infty} \end{matrix} \quad \frac{POLY_{Num}(x)\big|_{x \text{ near } x_0}}{POLY_{Den}(x)\big|_{x \text{ near } x_0}} \quad \begin{matrix} \xrightarrow{\text{i. Approximate}} \\ \xrightarrow{\text{i. Approximate}} \end{matrix} \quad \frac{Princ.TERM_{Num}(x)\big|_{x \text{ near } x_0} + [...]}{Princ.TERM_{Den}(x)\big|_{x \text{ near } x_0} + [...]}$$

ii. In order to divide $Princ.TERM_{Num(x)}\Big|_{x \text{ near } x_0}$, that is the principal term near x_0 of the *numerator* of RAT by $Princ.TERM_{Den(x)}\Big|_{x \text{ near } x_0}$, that is the principal term near x_0 of the *denominator* of RAT we use monomial division

$\boxed{\dfrac{ah^{+m}}{bh^{+n}} = \dfrac{a}{b}h^{+m\ominus+n}}$ where $+m \ominus +n$ can turn out positive, negative or 0

The resulting monomial is $Princ.TERM\,RAT(x)\Big|_{x \text{ near } x_0}$, that is the *principal term* of the rational function RAT near x_0.



However, $Princ.TERM\,RAT(x)\Big|_{x \text{ near } x_0}$ is useful only in four cases:

- When it is a constant term *and* what we want is the Height-sign,
- When it is a linear term *and* what we want is the Height-sign or the Slope-sign,
- When it is a square term,
- When it is a negative-exponent term.

■ The *long route* to $Princ.PART\ RAT(x)\ |_{x \text{ near } x_0}$:

i. In order to divide $POLY_{Num}(x)|_{x \text{ near } x_0}$ *by* $POLY_{Den}(x)|_{x \text{ near } x_0}$, we set up the division as a *long division*, that is $POLY_{Den}(x)|_{x \text{ near } x_0}$ dividing *into* $POLY_{Num}(x)|_{x \text{ near } x_0}$ and since these are polynomials in h, in order to be in order of descending sizes, they must be in order of ascending exponents.

ii. We approximate by stopping the long division as soon as we have the *principal part* that has the feature(s) we want:

iii. The difficulty will be that we will have to approximate at two different stages:

- While we localize both the numerator and the denominator,
- When we divide the approximate localization of the numerator by the approximate localization of the denominator

So, we will have to make sure that the approximations in the localizations of the numerator and the denominator do not interfere with the approximation in the division, that is that, as we divide, we do not want to bump into a [...] coming from having approximated the numerator and the denominator too much, that is before we can extract from the division the term that controls the wanted feature.

3. Which route we will take in each particular case will depend both on the *wanted feature(s)* near x_0 and so we will now look separately at how we get $Height\text{-sign}|_{x \text{ near } \infty}$, $Slope\text{-sign}|_{x \text{ near } x_0}$ and $Concavity\text{-sign}|_{x \text{ near } x_0}$

LOCAL ANALYSIS NEAR x_0

When the wanted features are to be found near x_0, the *rational degree* of the rational function does not tell us which of the *short route* or the *long route* will allow us to extract the term that controls the wanted feature.

17.2 Height-sign Near x_0

If all we want is the Height-sign, then we can always go the short route.

EXAMPLE 2. Let $SOUTH$ be the function specified by the global input-output rule

$$x \xrightarrow{SOUTH} SOUTH(x) = \frac{x^2+5x+6}{x^4-x^3-10x^2+x-15}$$

Find the height-sign of $SOUTH$ near $+2$

i. We localize both the numerator of $SOUTH$ and the denominator of $SOUTH$ near $+2$

$$\begin{aligned} h \xrightarrow{SOUTH_{+2}} SOUTH(+2+h) &= \left.\frac{x^2+5x+6}{x^4-x^3-10x^2+x-15}\right|_{x\leftarrow +2+h} \\ &= \frac{x^2+5x+6\big|_{x\leftarrow +2+h}}{x^4-x^3-10x^2+x-15\big|_{x\leftarrow +2+h}} \\ &= \frac{(+2+h)^2+5(+2+h)+6}{(+2+h)^4-(+2+h)^3-10(+2+h)^2+(+2+h)-15} \end{aligned}$$

ii. Since we want the local input-output rule that will give us the height-sign, we try to approximate *before* we divide:

$$
\begin{aligned}
&= \frac{\Big[(+2)^2 + 5\cdot(+2) + 6\Big] + [...]}{\Big[(+2)^4 - (+2)^3 - 10(+2)^2 + 2 - 15\Big] + [...]} \\
&= \frac{\Big[+4 + 10 + 6\Big] + [...]}{\Big[+16 - 8 - 40 + 2 - 15\Big] + [...]} \\
&= \frac{+20 + [...]}{-45 + [...]} \\
&= -\frac{20}{45} + [...]
\end{aligned}
$$

and since the approximate local input-output rule near $+2$ is

$$h \xrightarrow{SOUTH_{+2}} SOUTH(+2+h) = -\frac{20}{45} + [...]$$

and the local input-output rule includes the term that gives the Height-sign near $+2$

$$-\frac{20}{45}$$

we have:

$$\text{Height-sign } SOUTH \text{ near } +2 = (-.-)$$

EXAMPLE 3. Let $SOUTH$ be the function specified by the global input-output rule

$$x \xrightarrow{SOUTH} SOUTH(x) = \frac{x^2 + 5x + 6}{x^4 - x^3 - 10x^2 + x - 15}$$

Find the height-sign of $SOUTH$ near -3

i. We localize both the numerator of $SOUTH$ and the denominator of $SOUTH$ near -3

$$
\begin{aligned}
h \xrightarrow{SOUTH_{-3}} SOUTH(-3+h) &= \left.\frac{x^2 + 5x + 6}{x^4 - x^3 - 10x^2 + x - 15}\right|_{x\leftarrow -3+h} \\
&= \frac{x^2 + 5x + 6\big|_{x\leftarrow -3+h}}{x^4 - x^3 - 10x^2 + x - 15\big|_{x\leftarrow -3+h}} \\
&= \frac{(-3+h)^2 + 5(-3+h) + 6}{(-3+h)^4 - (-3+h)^3 - 10(-3+h)^2 + (-3+h) - 15}
\end{aligned}
$$

ii. Since we want the local input-output rule that will give us the height-sign, we try to approximate to the constant terms:

$$= \frac{\Big[(-3)^2 + 5\cdot(-3) + 6\Big] + [...]}{\Big[(-3)^4 - (-3)^3 - 10(-3)^2 - 3 - 15\Big] + [...]}$$
$$= \frac{\Big[+9 - 15 + 6\Big] + [...]}{\Big[+81 + 27 - 90 - 3 - 15\Big] + [...]}$$
$$= \frac{\Big[0\Big] + [...]}{\Big[0\Big] + [...]}$$

We cannot divide as we could get

$$= any\, size$$

iii. We therefore must approximate the localizations at least to h

$$= \frac{\Big[0\Big] + \Big[2\cdot(-3) + 5\Big]h + [...]}{\Big[0\Big] + \Big[+4(-3)^3 - 3(-3)^2 - 10\cdot 2(-3) + 1\Big]h + [...]}$$
$$= \frac{\Big[-6 + 5\Big]h + [...]}{\Big[-108 - 27 + 60 + 1\Big]h + [...]}$$
$$= \frac{\Big[-1\Big]h + [...]}{\Big[-74\Big]h + [...]}$$
$$= \frac{-h + [...]}{-74h + [...]}$$

We divide

$$= +\frac{1}{74} + [...]$$

and since the approximate local input-output rule near -3 is

$$h \xrightarrow{SOUTH_{-3}} SOUTH(-3+h) = +\frac{1}{74} + [...]$$

and the local input-output rule includes the term that gives the Height-sign near -3

$$+\frac{1}{74}$$

we have:

$$\text{Height-sign } SOUTH \text{ near } -3 = (+,+)$$

17.3 Slope-sign Near x_0

Example 4. Let $SOUTH$ be the function specified by the global input-output rule

$$x \xrightarrow{SOUTH} SOUTH(x) = \frac{x^2+5x+6}{x^4-x^3-10x^2+x-15}$$

find the slope-sign of $SOUTH$ near $+2$

i. We localize both the numerator of $SOUTH$ and the denominator of $SOUTH$ near $+2$ and since we want the approximate local input-output rule for the slope-sign, we will approximate to h:

$$+2+h \xrightarrow{SOUTH} SOUTH(+2+h) = \left.\frac{x^2+5x+6}{x^4-x^3-10x^2+x-15}\right|_{x\leftarrow +2+h}$$

$$= \frac{x^2+5x+6\big|_{x\leftarrow +2+h}}{x^4-x^3-10x^2+x-15\big|_{x\leftarrow +2+h}}$$

$$= \frac{(+2+h)^2+5(+2+h)+6}{(+2+h)^4-(+2+h)^3-10(+2+h)^2+(+2+h)-15}$$

$$= \frac{\Big[(+2)^2+5\cdot(+2)+6\Big]+\Big[2(+2)+5\Big]h+[...]}{\Big[(+2)^4-(+2)^3-10\cdot(+2)^2+(+2)-15\Big]+\Big[4(+2)^3-3(+2)^2-10\cdot 2(+2)+1\Big]h+[...]}$$

$$= \frac{\Big[+20\Big]+\Big[+9\Big]h+[...]}{\Big[-45\Big]+\Big[-19\Big]h+[...]}$$

ii. We set up the division with

$[-45]+[-19]h+[...]$ dividing *into* $[+20]+[+9]h+[...]$

that is:

$$\begin{array}{r|lll}
 & -\frac{20}{45} & -\frac{[9\cdot45]-[19\cdot20]}{45^2}h & +[...] \\ \hline
-45\ -19h\ +[...] & +20 & +9h & +[...] \\
 & +20 & +\frac{19\cdot20}{45}h & +[...] \\ \hline
 & 0 & +\frac{[9\cdot45]-[19\cdot20]}{45}h & +[...]
\end{array}$$

And since $[9\cdot45]-[19\cdot20] = 405-380 = +25$, the approximate local input-output rule near $+2$ is:

$$h \xrightarrow{SOUTH_{+2}} SOUTH(+2+h) = -\frac{20}{45} - \frac{25}{45^2}h + [...]$$

and the term that gives the slope-sign near $+2$ is

$$-\frac{25}{45^2}h$$

so that

$$\text{Slope-sign } SOUTH \text{ near } +2 = (\searrow.\searrow)$$

17.4 Concavity-sign Near x_0

EXAMPLE 5. Let $SOUTH$ be the function specified by the global input-output rule

$$x \xrightarrow{SOUTH} SOUTH(x) = \frac{x^2+5x+6}{x^4-x^3-10x^2+x-15}$$

find the concavity-sign of $SOUTH$ near $+2$

i. We localize both the numerator of $SOUTH$ and the denominator of $SOUTH$ near $+2$ and since we want the approximate local input-output rule for the slope-sign, we will approximate to h^2:

$$+2+h \xrightarrow{SOUTH} SOUTH(+2+h) = \left.\frac{x^2+5x+6}{x^4-x^3-10x^2+x-15}\right|_{x\leftarrow +2+h}$$

$$= \frac{x^2+5x+6\big|_{x\leftarrow +2+h}}{x^4-x^3-10x^2+x-15\big|_{x\leftarrow +2+h}}$$

$$= \frac{(+2+h)^2+5(+2+h)+6}{(+2+h)^4-(+2+h)^3-10(+2+h)^2+(+2+h)-15}$$

$$= \frac{\Big[(+2)^2+5\cdot(+2)+6\Big]+\Big[2(+2)+5\Big]h+\Big[1\Big]h^2}{\Big[(+2)^4-(+2)^3-10\cdot(+2)^2+(+2)-15\Big]+\Big[4(+2)^3-3(+2)^2-10\cdot 2(+2)+1\Big]h+\Big[6(+2)^2-3(+2)-10\Big]h^2+[...]}$$

$$= \frac{\Big[+20\Big]+\Big[+9\Big]h+\Big[1\Big]h^2}{\Big[-45\Big]+\Big[-19\Big]h+\Big[8\Big]h^2+[...]}$$

ii. We set up the division with

$$-45+-19h+8h^2+[...] \quad \text{dividing } \textit{into} \quad +20+9h+h^2$$

but carry it out *latin style* (that is, we *write* the *result* of the multiplication as it comes out instead of the *opposite of the result*.)

$$\begin{array}{r|lllll}
 & -\frac{20}{45} & -\frac{[9\cdot45]-[19\cdot20]}{45^2}h & & -\left[\frac{+45[45-8\cdot20]-19[[9\cdot45]-[19\cdot20]]}{45^3}\right]h^2 & +[...] \\
\hline
-45\ -19h\ +8h^2\ +[...] & +20 & +9h & & +h^2 & \\
 & +20 & +\frac{19\cdot20}{45}h & & -\frac{8\cdot20}{45}h^2 & +[...] \\
\hline
 & 0 & +\frac{[9\cdot45]-[19\cdot20]}{45}h & & +\frac{45-8\cdot20}{45}h^2 & +[...] \\
 & & +\frac{[9\cdot45]-[19\cdot20]}{45^2}h & & +19\frac{[9\cdot45]-[19\cdot20]}{45^2}h^2 & +[...] \\
\hline
 & & +0h & + & \left[\frac{+45[45-8\cdot20]-19[[9\cdot45]-[19\cdot20]]}{45^2}\right]h^2 & +[...]
\end{array}$$

And since $\frac{+45[45-8\cdot20]-19[[9\cdot45]-[19\cdot20]]}{45^2} = -\frac{2401}{45^2}$, the local input-output rule near $+2$ is:

$$h \xrightarrow{SOUTH_{+2}} SOUTH(+2+h) = -\frac{20}{45}-\frac{25}{45^2}h-\frac{2401}{45^2}h^2+[...]$$

and the term that gives the concavity-sign near $+2$ is

$$-\frac{2401}{45^2}h^2$$

so that

$$\text{Concavity-sign } SOUTH \text{ near } +2 = (\cap, \cap)$$

17.5 Local Graph Near x_0

EXAMPLE 6. Let $SOUTH$ be the function specified by the global input-output rule

$$x \xrightarrow{SOUTH} SOUTH(x) = \frac{x^2 + 5x + 6}{x^4 - x^3 - 10x^2 + x - 15}$$

find the local graph of $SOUTH$ near $+2$

Since, in order to get the local graph near $+2$ we need all three features near $+2$, height-sign, slope-sign and concavity-sign, we need to get the approximate local input-output rule as we did in the previous example:

$$h \xrightarrow{SOUTH_{+2}} SOUTH(+2 + h) = -\frac{20}{45} - \frac{25}{45^2}h - \frac{2401}{45^2}h^2 + [...]$$

from which we get: