Chapter 18

Rational Functions: Global Analysis

Contrary to what we were able to do with polynomial functions, with rational functions we will *not* be able to establish global theorems. Of course, we did not really establish global theorems for *all* polynomial functions either but only for polynomial functions *of a given degree*, 0, 1, 2 and 3. But, in the case of rational functions, even the *rational degree* will not separate rational functions into kinds that we can establish global theorems for inasmuch as even rational functions with a given rational degree can be very diverse.

So, what we will do here is to focus on how to get global information about *any given rational function*.

18.1 The Essential Question

Given a *rational function*, as with any function, the *offscreen graph* will consist:

- certainly of the local graph near ∞. This is because, as soon as the *input* is *large*, the graph point is going to be left or right of the screen and therefore *offscreen* regardless of the size of the *output*,

- possibly of the local graph(s) near certain *bounded input(s)*. This is because, in case the outputs for inputs near certain bounded inputs are *large*, the graph points will then be above or below the screen and therefore *offscreen* even though the inputs are *bounded.*

So, as always, we will need to ascertain whether

- There might be *bounded inputs* for which nearby inputs will have a *large* output ,

or, as was the case with all polynomial functions,

- The outputs for any *bounded input* are themselves necessarily *bounded*

In other words, in order to get the *offscreen graph*, we must begin by asking the **Essential Question**:

> - Do all *bounded inputs* have *bounded outputs*
>
> or
>
> - Is there one (or more) *bounded input* which is an ∞-height input, that is, a *bounded input* whose nearby inputs have *infinite outputs*?

And, indeed, we will find that there are two kinds of rational functions:

- rational functions that *do* have ∞-height input(s)
- rational function that *do not* have any ∞-height input as was the case with power functions and polynomial functions.

18.2 Locating ∞-Height Inputs

However, given a rational function, not only will we need to know whether or not there *exists* ∞-height input(s), if there are any, we will also have to *locate* the ∞-height inputs, if any, because we will need to get the local graph near these ∞-height input(s). More precisely:

1. Given a rational function *RAT* specified by a global input-output rule

$$x \xrightarrow{RAT} RAT(x) = \frac{NUMERATOR_{RAT}(x)}{DENOMINATOR_{RAT}(x)}$$

we want to find whether or not there can be a *bounded input* x_0 such that the outputs for *nearby* inputs, $x_0 + h$, are *large.* In other words, we want to know if there can be x_0 such that

$$h \xrightarrow{RAT} RAT(x)|_{x \leftarrow x_0+h} = large$$

But we have

$$RAT(x)|_{x \leftarrow x_0+h} = \left.\frac{NUMERATOR_{RAT}(x)}{DENOMINATOR_{RAT}(x)}\right|_{x \leftarrow x_0+h}$$
$$= \frac{NUMERATOR_{RAT}(x)|_{x \leftarrow x_0+h}}{DENOMINATOR_{RAT}(x)|_{x \leftarrow x_0+h}}$$
$$= \frac{NUMERATOR_{RAT}(x_0+h)}{DENOMINATOR_{RAT}(x_0+h)}$$

So, what we want to know is if there can be an x_0 for which
$\frac{NUMERATOR_{RAT}(x_0+h)}{DENOMINATOR_{RAT}(x_0+h)} = large$

2. Since it is a *fraction* that we want to be *large*, we will use the **Division Size Theorem** from **Chapter 2**:

THEOREM 2 (**Division Size**)

$$\frac{large}{large} = any\ size \qquad \frac{large}{medium} = \boldsymbol{large} \qquad \frac{large}{small} = \boldsymbol{large}$$
$$\frac{medium}{large} = small \qquad \frac{medium}{medium} = medium \qquad \frac{medium}{small} = \boldsymbol{large}$$
$$\frac{small}{large} = small \qquad \frac{small}{medium} = small \qquad \frac{small}{small} = any\ size$$

There are thus two ways that a fraction can be *large*:
- When the numerator is *large*
- When the denominator is *small*

In each case, though, we need to make sure of the other side of the fraction. So, rather than look at the size of both the numerator and the denominator at the same time, we will look separately at:
- The first *row*, that is when the *numerator* of the fraction is *large*

$$\frac{large}{large} = any\ size \qquad \frac{large}{medium} = \boldsymbol{large} \qquad \frac{large}{small} = \boldsymbol{large}$$
$$\frac{medium}{large} = small \qquad \frac{medium}{medium} = medium \qquad \frac{medium}{small} = \boldsymbol{large}$$
$$\frac{small}{large} = small \qquad \frac{small}{medium} = small \qquad \frac{small}{small} = any\ size$$

  because in that case all we will then have to do is to make sure that the *denominator* is *not large* too.
- The last *column*, that is when the *denominator* of the fraction is *small*.

possible ∞-height input

$$\frac{large}{large} = any\ size \qquad \frac{large}{medium} = \boldsymbol{large} \qquad \frac{large}{small} = \boldsymbol{large}$$

$$\frac{medium}{large} = small \qquad \frac{medium}{medium} = medium \qquad \frac{medium}{small} = \boldsymbol{large}$$

$$\frac{small}{large} = small \qquad \frac{small}{medium} = small \qquad \frac{small}{small} = any\ size$$

because in that case all we will then have to do is to make sure that the numerator is *not small* too.

3. We now deal with $\frac{NUMERATOR_{RAT}(x_0+h)}{DENOMINATOR_{RAT}(x_0+h)}$, looking separately at the numerator and the denominator:

- Since the *numerator*, $NUMERATOR_{RAT}(x_0 + h)$, is the output of a *polynomial function*, namely

$$x \xrightarrow{NUMERATOR_{RAT}} NUMERATOR_{RAT}(x)$$

and since we have seen that *the only way* the outputs of a *polynomial function* can be *large* is when the inputs are themselves *large*, *there is no way* that $NUMERATOR_{RAT}(x_0+h))$ could be *large* for inputs that are *bounded*. So there is no way that the output of RAT could be large for *bounded* inputs that make the *numerator* large and we need not look any further.

- Since the *denominator*, $DENOMINATOR_{RAT}(x_0 + h)$, is the output of the *polynomial function*

$$x \xrightarrow{DENOMINATOR_{RAT}} DENOMINATOR_{RAT}(x)$$

and since we have seen that polynomial functions *can* have *small* outputs if they have 0-height inputs and the inputs are near the 0-height inputs, $DENOMINATOR_{RAT}(x_0+h)$ *can* be *small* for certain bounded inputs and thus so can $\frac{NUMERATOR_{RAT}(x_0+h)}{DENOMINATOR_{RAT}(x_0+h)}$. However, we will then have to make sure that $NUMERATOR_{RAT}(x_0 + h)$, is *not small* too near these bounded inputs, that is we will have to make sure that x_0 does *not* turn out to be a 0-height input for $NUMERATOR_{RAT}$ as well as for $DENOMINATOR_{RAT}$ so as not to be in the case:

$$\frac{small}{small} = any\ size$$

We will thus refer to a 0-height input for $DENOMINATOR_{RAT}$ as only a **possible ∞-height input** for RAT

Altogether, then, we have:

THEOREM 1 (Possible ∞-height Input). *The* 0*-height inputs of the* denominator *of a rational function, if any, are the only* possible ∞-height inputs *for the rational function.*

4. However, this happens to be one of these very rare situations in which there *is* "an easier way": After we have located the 0-height inputs for $DENOMINATOR_{RAT}$, instead of first making sure that they are not also 0-height inputs for $NUMERATOR_{RAT}$, we will gamble and just get the local input-output rule near each one of the 0-height inputs for $DENOMINATOR_{RAT}$. Then,

- If the local input-ouput rule turns out to start with a *negative-exponent power function*, then we will have determined that x_0 *is* an ∞-height input for RAT and the payoff will be that we will now get the local graph near x_0 for free.
- If the local input-ouput rule turns out *not* to start with a *negative-exponent power function*, then we will have determined that x_0 is *not* a ∞-height input for RAT after all and our loss will be that we will probably have no further use for the local input-output rule.

Overall, then, we will use the following two steps:

Step i. *Locate* the 0-height inputs for the *denominator*, $DENOMINATOR_{RAT}(x)$, by solving the equation

$$DENOMINATOR_{RAT}(x) = 0$$

Step ii. Compute the *local input-output rule* near each one of the 0-height inputs for the *denominator*, if any.

The advantage is that we need not even refer to the **Division Size Theorem**: once we have a possible ∞-height input, we just get the local input-output rule near that possible ∞-height input, "for the better or for the worse".

EXAMPLE 1. Let $COUGH$ be the function specified by the global input-output rule

$$x \xrightarrow{COUGH} COUGH(x) = \frac{x^4 - x^3 - 10x^2 + x - 15}{x^2 + 5x + 6}$$

locate the ∞-height input(s) of $COUGH$, if any.

Step i. The possible ∞-height input(s) of $COUGH$ are the 0-height input(s) of $DENOMINATOR_{COUGH}(x)$, that is the solution(s), if any, of the equation

$$x^2 + 5x + 6 = 0$$

In general, solving an equation may or may not possible but in this case, the equation is a *quadratic* one and we have learned how to do this in **Chapter 12**. One way or the other, we find that there are two solutions:

$$-3, -2$$

which are the *possible* ∞*-height inputs* of the rational function $COUGH$.

Step ii. We compute the local input-output rules near -3 and near -2:

- Near -3:

$$h \xrightarrow{COUGH_{\text{near } -3}} COUGH(-3+h) = \left.\frac{x^4 - x^3 - 10x^2 + x - 15}{x^2 + 5x + 6}\right|_{x \leftarrow -3+h}$$
$$= \frac{x^4 - x^3 - 10x^2 + x - 15|_{x \leftarrow -3+h}}{x^2 + 5x + 6|_{x \leftarrow -3+h}}$$
$$= \frac{(-3+h)^4 - (-3+h)^3 - 10(-3+h)^2 + (-3+h) - 15}{(-3+h)^2 + 5(-3+h) + 6}$$

We try to approximate to the constant terms:

$$= \frac{(-3)^4 + [...] - (-3)^3 + [...] - 10(-3)^2 + [...] - 3 + [...] - 15}{(-3)^2 + [...] + 5(-3) + [...] + 6}$$
$$= \frac{+81 + 27 - 90 - 3 - 15 + [...]}{+9 - 15 + 6 + [...]}$$
$$= \frac{0 + [...]}{0 + [...]}$$
$$= \frac{[...]}{[...]}$$
$$= \textit{any size}$$

So we must go back and try to approximate to the linear terms, ignoring the constant terms since we just saw that they add up to 0 both in the numerator and the denominator:

$$= \frac{4(-3)^3h + [...] - 3(-3)^2h + [...] - 10 \cdot 2(-3)h + [...] + h}{2 \cdot (-3)h + [...] + 5h}$$
$$= \frac{-108h + [...] - 27h + [...] + 60h + [...] + h}{-6h + [...] + 5h}$$
$$= \frac{-74h + [...]}{-h + [...]}$$
$$= +74 + [...]$$

so that -3 is *not* an ∞-heigth input

- Near -2:

$$h \xrightarrow{COUGH_{\text{near } -2}} COUGH(-2+h) = \left.\frac{x^4 - x^3 - 10x^2 + x - 15}{x^2 + 5x + 6}\right|_{x \leftarrow -2+h}$$
$$= \frac{x^4 - x^3 - 10x^2 + x - 15|_{x \leftarrow -2+h}}{x^2 + 5x + 6|_{x \leftarrow -2+h}}$$
$$= \frac{(-2+h)^4 - (-2+h)^3 - 10(-2+h)^2 + (-2+h) - 15}{(-2+h)^2 + 5(-2+h) + 6}$$

We try to approximate to the constant terms:

$$= \frac{(-2)^4 + [...] - (-2)^3 + [...] - 10(-2)^2 + [...] - 2 + [...] - 15}{(-2)^2 + [...] + 5(-2) + [...] + 6}$$
$$= \frac{+16 + 8 - 40 - 2 - 15 + [...]}{+4 - 10 + 6 + [...]}$$
$$= \frac{-33 + [...]}{0 + [...]}$$
$$= \frac{-33}{[...]}$$
$$= large$$

So -2 is an ∞-height input for $COUGH$ and we need only find exactly how small [...] is to get the local input-output rule near -2

$$= \frac{-33 + [...]}{2 \cdot (-2)h + [...] + 5h}$$
$$= \frac{-33 + [...]}{h + [...]}$$
$$= -33h^{-1} + [...]$$

18.3 Offscreen Graph

Once the Essential Question has been answered, and if we do not already have the local input-output rule near each one of the ∞-height inputs, we need to get them and the corresponding local graphs so that we can then join them smoothly to get the offscreen graph.

Altogether, given a rational function RAT the procedure to obtain the *offscreen graph* is therefore:

i. Get the approximate input-output rule near ∞ and the local graph near ∞

ii. Answer the **Essential Question** and locate the ∞ input(s), if any,
iii. Find the local input-output rule and then the local graphs near each ∞-height inputs

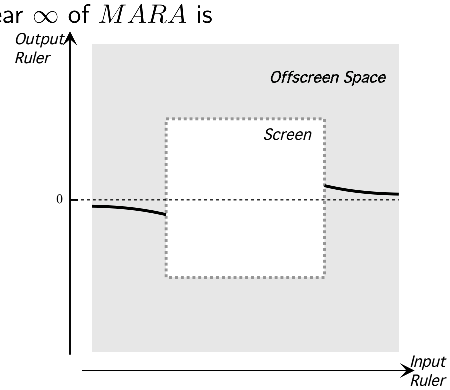
EXAMPLE 2. Let $MARA$ be the function specified by the global input-output rule

$$x \xrightarrow{MARA} MARA(x) = \frac{x - 15}{x^2 + 5x + 7}$$

Find the offscreen graph.
i. We get the local approximation near ∞:

$$\text{Near } \infty,\ x \xrightarrow{MARA} MARA(x) = \frac{x + [...]}{x^2 + [...]}$$
$$= +x^{-1} + [...]$$

and the local graph near ∞ of $MARA$ is



ii. We locate the ∞-height inputs, if any. The possible ∞-height input(s) of $MARA$ are the 0-height input(s) of $DENOMINATOR_{MARA}(x)$, that is the solution(s), if any, of the equation

$$x^2 + 5x + 7 = 0$$

In general, solving an equation may or may not possible but in this case, the equation is a *quadratic* one and we have learned how to do this in **Chapter 12**. One way or the other, we find that there are no solution. So, the function $MARA$ has no ∞-height input.
iii. The *offscreen graph* therefore consists of only the local graph near ∞.

18.4 Feature-sign Change Inputs

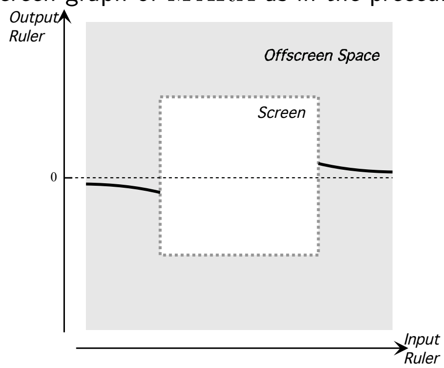
Given a rational function, in order to get the feature-sign change input(s), if any, we need only get the outlying graph and then we proceed as in **Chapter 3** so we need only give an example.

EXAMPLE 3. Let $MARA$ be the function specified by the global input-output rule

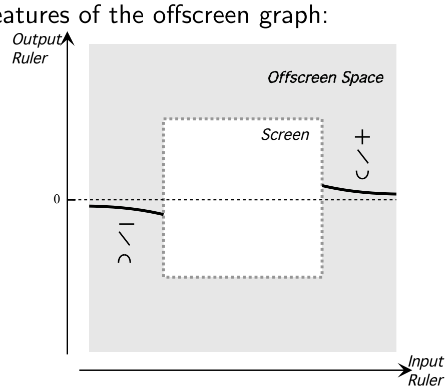
$$x \xrightarrow{MARA} MARA(x) = \frac{x-15}{x^2+5x+7}$$

Find the feature-sign change inputs of $MARA$, if any.

i. We find the offscreen graph of $MARA$ as in the preceding example:



ii. We mark the features of the offscreen graph:



iii. Therefore:

- there must be at least one height-sign change input,
- there does not have to be a slope-sign change input
- there must be at least one concavity-sign change input,

18.5 Global Graph

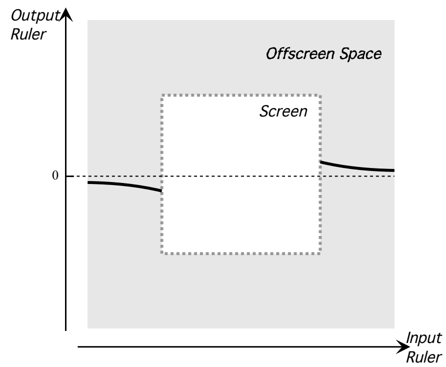
Given a rational function, in order to get the essential global graph, we need only get the outlying graph and then we join smoothly so we need only give an example.

EXAMPLE 4. Let $MARA$ be the function specified by the global input-output rule

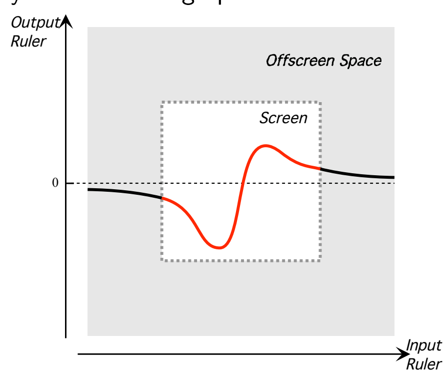
$$x \xrightarrow{MARA} MARA(x) = \frac{x-15}{x^2+5x+7}$$

Find the feature-sign change inputs of $MARA$, if any.

i. We find the offscreen graph of $MARA$ as in the preceding example:

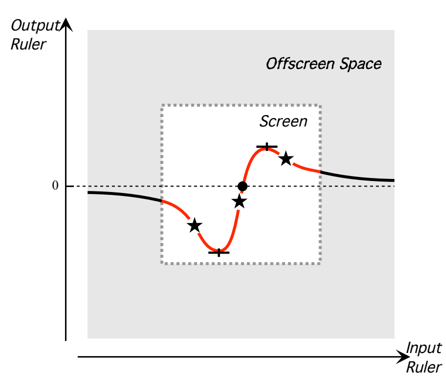


ii. We join smoothly the offscreen graph:



iii. Observe that, in fact,

- there must be at least one height-sign change input,
- there must be at least *two* slope-sign change inputs
- there must be at least *three* concavity-sign change input,



18.6 Locating 0-Height Inputs

Locating the 0-height inputs of a given rational function is pretty much the mirror image of what we did to locate its ∞-height inputs. More precisely:

1. Given a rational function RAT specified by a global input-output rule

$$x \xrightarrow{\quad RAT \quad} RAT(x) = \frac{NUMERATOR_{RAT}(x)}{DENOMINATOR_{RAT}(x)}$$

we want to find whether or not there can be a *bounded input* x_0 such that the outputs for *nearby* inputs, $x_0 + h$, are *small*. In other words, we want to know if there can be x_0 such that

$$h \xrightarrow{\quad RAT \quad} RAT(x)|_{x \leftarrow x_0+h} = small$$

But we have

$$\begin{aligned} RAT(x)|_{x \leftarrow x_0+h} &= \left.\frac{NUMERATOR_{RAT}(x)}{DENOMINATOR_{RAT}(x)}\right|_{x \leftarrow x_0+h} \\ &= \frac{NUMERATOR_{RAT}(x)|_{x \leftarrow x_0+h}}{DENOMINATOR_{RAT}(x)|_{x \leftarrow x_0+h}} \\ &= \frac{NUMERATOR_{RAT}(x_0+h)}{DENOMINATOR_{RAT}(x_0+h)} \end{aligned}$$

So, what we want to know is if there can be an x_0 for which

$$\frac{NUMERATOR_{RAT}(x_0+h)}{DENOMINATOR_{RAT}(x_0+h)} = small$$

2. Since it is a *fraction* that we want to be *small*, we will use the **Division Size Theorem** from **Chapter 2**:

THEOREM 2 (**Division Size**)

$$\frac{large}{large} = any\ size \qquad \frac{large}{medium} = large \qquad \frac{large}{small} = large$$

$$\frac{medium}{large} = \boldsymbol{small} \qquad \frac{medium}{medium} = medium \qquad \frac{medium}{small} = large$$

$$\frac{small}{large} = \boldsymbol{small} \qquad \frac{small}{medium} = \boldsymbol{small} \qquad \frac{small}{small} = any\ size$$

There are thus two ways that a fraction can be *small*:

- When the numerator is *small*
- When the denominator is *large*

In each case, though, we need to make sure of the other side of the fraction. So, rather than look at the size of both the numerator and the denominator at the same time, we will look separately at:

- The third *row*, that is when the *numerator* of the fraction is *small*

$$\frac{large}{large} = any\ size \qquad \frac{large}{medium} = large \qquad \frac{large}{small} = large$$

$$\frac{medium}{large} = small \qquad \frac{medium}{medium} = medium \qquad \frac{medium}{small} = large$$

$$\frac{small}{large} = \boldsymbol{small} \qquad \frac{small}{medium} = \boldsymbol{small} \qquad \frac{small}{small} = any\ size$$

  because in that case all we will then have to do is to make sure that the *denominator* is *not small* too.

- The first *column*, that is when the *denominator* of the fraction is *large*.

$$\frac{large}{large} = any\ size \qquad \frac{large}{medium} = large \qquad \frac{large}{small} = large$$

$$\frac{medium}{large} = \boldsymbol{small} \qquad \frac{medium}{medium} = medium \qquad \frac{medium}{small} = large$$

$$\frac{small}{large} = \boldsymbol{small} \qquad \frac{small}{medium} = \boldsymbol{small} \qquad \frac{small}{small} = any\ size$$

  because in that case all we will then have to do is to make sure that the numerator is *not large* too.

3. We now deal with $\frac{NUMERATOR_{RAT}(x_0+h)}{DENOMINATOR_{RAT}(x_0+h)}$, looking separately at the numerator and the denominator:

- Since the *numerator*, $NUMERATOR_{RAT}(x_0 + h)$, is the output of a *polynomial function*, namely

$$x \xrightarrow{NUMERATOR_{RAT}} NUMERATOR_{RAT}(x)$$

  and since we have seen that polynomial functions *can* have *small* outputs if they have 0-height inputs and the inputs are near the 0-height inputs, $NUMERATOR_{RAT}(x_0 + h)$ *can* be *small* for certain bounded inputs and thus so can $\frac{NUMERATOR_{RAT}(x_0+h)}{DENOMINATOR_{RAT}(x_0+h)}$. However, we will then have to make sure that $DENOMINATOR_{RAT}(x_0+h)$, is *not small* too near these bounded inputs, that is we will have to make sure that x_0 does *not* turn out to be a 0-height input for $DENOMINATOR_{RAT}$ as well as for $NUMERATOR_{RAT}$ so as not to be in the case:

$$\frac{small}{small} = any\ size$$

We will thus refer to a 0-height input for $NUMERATOR_{RAT}$ as only a **possible 0-height input** for RAT. possible 0-height input

- Since the *denominator*, $DENOMINATOR_{RAT}(x_0 + h)$, is the output of a *polynomial function*, namely

$$x \xrightarrow{DENOMINATOR_{RAT}} DENOMINATOR_{RAT}(x)$$

  and since we have seen that *the only way* the outputs of a *polynomial function* can be *large* is when the inputs are themselves *large*, *there is no way* that $DENOMINATOR_{RAT}(x_0 + h))$ could be *large* for inputs that are *bounded.* So there is no way that the output of RAT could be *small* for *bounded* inputs that make the *denominator* large and we need not look any further.

Altogether, then, we have:

THEOREM 2 (Possible 0-height Input). *The 0-height inputs of the* numerator *of a rational function, if any, are the only* possible 0-height inputs *for the rational function.*

4. However, this happens to be one of these very rare situations in which there *is* "an easier way": After we have located the 0-height inputs for $NUMERATOR_{RAT}$, instead of first making sure that they are not also 0-height inputs for $DENOMINATOR_{RAT}$, we will gamble and just get the local input-output rule near each one of the 0-height inputs for $NUMERATOR_{RAT}$. Then,

- If the local input-ouput rule turns out to start with a *positive-exponent power function*, then we will have determined that x_0 *is* a 0-height input for RAT and the payoff will be that we will now get the local graph near x_0 for free.
- If the local input-ouput rule turns out to start with a 0-*exponent power function* or a *negative-exponent power function*, then we will have determined that x_0 is *not* a 0-height input for RAT after all and our loss will be that we will probably have no further use for the local input-output rule.

Overall, then, we will use the following two steps:

Step i. *Locate* the 0-height inputs for the *numerator*, $NUMERATOR_{RAT}(x)$, by solving the equation

$$NUMERATOR_{RAT}(x) = 0$$

Step ii. Compute the *local input-output rule* near each one of the 0-height inputs for the *numerator*, if any.

The advantage is that we need not even refer to the **Division Size Theorem**: once we have a possible 0-height input, we just get the local input-output rule near that possible 0-height input, "for the better or for the worse".

EXAMPLE 5. Let $TARA$ be the function specified by the global input-output rule

$$x \xrightarrow{TARA} TARA(x) = \frac{x^3 - 8}{x^2 + 3x - 10}$$

locate the 0-height input(s) if any.

Step i. The possible 0-height input(s) of $TARA$ are the 0-height input(s) of $NUMERATOR_{TARA}(x)$, that is the solution(s), if any, of the equation

$$x^3 - 8 = 0$$

In general, solving an equation may or may not possible and in this case, the equation is a *cubic* one. Still, here it is a very incomplete one and we can see that the solution is $+2$ which is the possible 0-height input of the rational function $TARA$.

Step ii. We compute the local input-output rule near $+2$.

$$\begin{aligned} h \xrightarrow{TARA_{\text{ near } -3}} TARA(+2+h) &= \left.\frac{x^3 - 8}{x^2 + 3x - 10}\right|_{x \leftarrow +2+h} \\ &= \frac{x^3 - 8\big|_{x \leftarrow +2+h}}{x^2 + 3x - 10\big|_{x \leftarrow +2+h}} \\ &= \frac{(+2+h)^3 - 8}{(+2+h)^2 + 3(+2+h) - 10} \end{aligned}$$

We try to approximate to the constant terms:

$$\begin{aligned} &= \frac{(+2)^3 + [...] - 8}{(+2)^2 + [...] + 3(+2) + [...] - 10} \\ &= \frac{+8 - 8 + [...]}{+4 + 6 - 10 + [...]} \\ &= \frac{0 + [...]}{0 + [...]} = \frac{[...]}{[...]} = \textit{any size} \end{aligned}$$

So we must go back and approximate to the linear terms, ignoring the constant terms since we just saw that they add up to 0 both in the numerator and the denominator:

$$\begin{aligned} &= \frac{3(+2)^2 h + [...]}{2(+2)h + [...] + 3h} \\ &= \frac{+12h + [...]}{+4h + [...] + 3h} = \frac{+12h + [...]}{+7h + [...]} \\ &= +\frac{12}{7} + [...] \end{aligned}$$

and, since $+\frac{12}{7} \neq 0$, $+2$ is *not* an 0-heigth input for $TARA$.