Chapter 2
Towards Local Analysis

Size Of Numbers, 23 – Qualitative Rulers, 27 – Defining large, small And bounded, 28 – Computing With large, small And bounded, 30 – Infinity, 32 – Localization, 35.

Given that we will want to look at functions from a qualitative viewpoint, it should make sense that we need to look at numbers also from a qualitative viewpoint which is a viewpoint that is somewhat different—and that we need to distinguish—from the more common quantitative view which is the view usually taken in high school.

2.1 Size Of Numbers

A signed number carries two pieces of information: its sign, which is + or −, and its size, which is a plain number, namely an “unsigned number”.

**Example 1.** Given the signed number +17.43,
- *Sign* + 17.43 = +.
- *Size* + 17.43 = 17.43 (which is a plain number.)

**Example 2.** Given the signed number −25000,
- *Sign* − 25000 = −.
- *Size* − 25000 = 25000 (which is a plain number.)

**Note.** Instead of “size”, textbooks mostly use the term “absolute value” but “numerical value”, “modulus”, “norm”, are also to be found.

1. More precisely, from the graphic viewpoint:
- Once we subscribe to the usual agreement that rulers are drawn in such a way that
– *positive* numbers are to the *right* of 0
– *negative* numbers are to the *left* of 0

the *sign* of a signed number indicates *which side* of 0 the signed number is on a *quantitative ruler*.

**Example 3.**
– Since $\text{Sign} \ -5 = -, \text{the point that represents} \ -5 \text{is left of} \ 0.$
– Since $\text{Sign} \ +5 = +, \text{the point that represents} \ +5 \text{is right of} \ 0.$

- The *size* of a signed number indicates *how far* from 0 the signed number is.

**Example 4.** $-5$ and $+5$ are the same *size*, namely 5, so they are at the same distance from 0:

- $-5$ is 5 away from 0
- $+5$ is 5 away from 0

2. The next thing is to deal with the fact that there are *two* ways in which we can compare *signed* numbers and that we will have to use different phrases to make sure of which one we mean.

a. The usual way to compare two *signed* numbers is just to extend the order for *plain* numbers to *signed* numbers. When comparing signed number that way, we will use the phrases *absolutely-larger* and *absolutely-smaller*.

- Graphically, the more to the right, the absolutely-larger and the more to the left, the absolutely-smaller.
- Computationally:
  - If two signed numbers have *different signs*, then
    * the signed number whose sign is $-$ is **absolutely-smaller** than the signed number whose sign is $+$.
    * the signed number whose sign is $+$ is **absolutely-larger** than the signed number whose sign is $-$.

1 Educologists are bound to object that, say, “absolutely-larger” is likely to evoke “larger in absolute value”. While they may be correct, i. it is not a foregone conclusion, ii. we will use “size” in place of “absolute value” and, iii. the distinction between “absolutely-larger” and “larger in size” will be necessary in the next chapter and there does not seem to be much of an alternative.
2.1. SIZE OF NUMBERS

– If two signed numbers have the same signs, then
  * If they both have the + sign, then the signed number with the smaller size is absolutely-smaller than the signed number with the larger size.
  * If they both have the – sign, then the signed number with the smaller size is absolutely-larger than the signed number with the larger size.

EXAMPLE 5.

−7 is absolutely-smaller than −3

and

+3 is absolutely-smaller than +7

Graphically, this corresponds to the fact that the plot point for −7 is left of the plot point for −3

the same way that the plot point for +3 is left of the plot point for +7

b. But we can also compare two signed numbers in terms of just their “sizes” and ignore their signs.

■ Computationally:
  – A first signed number is larger-in-size than a second signed number if the size of the first signed number is larger than the size of the second signed number.
  – A first signed number is smaller-in-size than a second signed number if the size of the first signed number is smaller than the size of the second signed number.

■ Graphically

2It is not at all clear why Educologists refuse to make the distinction since in common parlance a $2000 debt is larger-in-size than a $1000 debt. Could it be because explaining this practice would require them to distinguish between

“−2000 Dollars” as written in Mathematics where −2000 is a (signed) numerator and Dollars is a (plain) denominator,

and

“2000 Debit Dollars” as written in Accounting where 2000 is a (plain) numerator and Debit Dollars is a (signed) denominator (written on top of the column), whereas they summarily dismiss the very idea of number-phrase, that is of numerator-denominator pair? (Some have even been known to invoke a “dangerous” confusion with fractions without noticing that a fraction is exactly that: a number-phrase.)
A first signed number is larger-in-size than a second signed number if the first signed number is further away from 0 than the second signed number.

A first signed number is smaller-in-size than a second signed number if the first signed number is closer to 0 than the second signed number.

**Example 6.**

- +3 is **smaller-in-size** than +7
  This is because the size of +3 is 3 which is smaller than the size of +7 which is 7.
  Graphically, this corresponds to the fact that the tick-mark for +3 is closer from the origin 0 than the tick-mark for +7.

- +3 is **smaller-in-size** than −7
  This is because the size of +3 is 3 which is smaller than the size of −7 which is 7.
  Graphically, this corresponds to the fact that the tick-mark for +3 is closer from the origin 0 than the tick-mark for −7.

- −3 is **smaller-in-size** than +7
  This is because the size of −3 is 3 which is smaller than the size of +7 which is 7.
  Graphically, this corresponds to the fact that the tick-mark for −3 is closer from the origin 0 than the tick-mark for +7.

- −3 is **smaller-in-size** than −7
  This is because the size of −3 is 3 which is smaller than the size of −7 which is 7.
  Graphically, this corresponds to the fact that the tick-mark for −3 is closer from the origin 0 than the tick-mark for −7.

**c.** For the sake of completion, we note that:
2.2. QUALITATIVE RULERS

- A number and the opposite of that number are absolutely-different but they are equal-in-size,
- When a number is either larger-in-size or smaller-in-size than another number, the two numbers are different-in-size.

2.2 Qualitative Rulers

When picturing numbers from the qualitative viewpoint, we will use what we will call qualitative rulers and we will now examine what we can retain from the concept of quantitative ruler when we move to the concept of qualitative ruler.

Indeed, since there will be mostly no tick-marks on a qualitative ruler this would appear to create two kinds of difficulties inasmuch as both the concept of extent and the concept of resolution would then seem to disappear along with the tick-marks.

1. We begin with what has to do with the extent. Even if we may not know what the numbers are that specify the extent of a ruler, a ruler has to have an extent as, else, it would be nothing more than a straightedge and so we will use qualitative rulers that is rulers that look like

The parentheses mark whereabouts on the ruler the last tick-marks of the extent would be if the ruler were a quantitative ruler.

Now, even though we do not know what the numbers that specify the extent of a qualitative ruler are, it still remains that any given number will fall either within the extent of the qualitative ruler or beyond the extent of the qualitative ruler.

- Numbers that fall within the extent of the qualitative ruler will continue to be called bounded numbers. The reason is that, as we saw with quantitative rulers, any bunch of number can be viewed as a bunch of bounded numbers so that it will not be a problem that the numbers that specify the extent are undisclosed.
- Numbers that fall beyond the extent of the given qualitative ruler will continue to be called offscreen numbers.

**Example 7.** Given the qualitative ruler

we have

opposite absolutely-different equal-in-size different-in-size qualitative ruler, qualitative beyond the extent bounded numbers offscreen numbers
2. We now turn to issues that have to do with the resolution. Even if we may not know what the finite numbers are, that is which numbers are tick-marked on the ruler, to make sense, there has to be at least some number that is tick-marked if only the number that we happen to be interested in at the moment, call it $x_0$. And, if not, we can always do so.

Usually, though, we will be interested in a neighborhood of $x_0$ but since there will usually not be any other tick-mark, we cannot say that the neighborhood of $x_0$ extends to the nearest half-marks. So, on qualitative rulers, we will just use a “small” highlighted piece of the ruler between parentheses to indicate the neighborhood of $x_0$.

2.3 Defining $large$, small And bounded

We now come to something that is absolutely and totally crucial in mathematics and which will be at the heart of all that we will be doing from now on.

1. While the concepts of absolutely-larger and larger-in-size are both quite clear, the apparently simpler concept of large is in fact not at all simple to pin down. Before anything else, though, let us say that, from now on, by large we will mean large-in-size, that is we will be ignoring the sign.

a. Up to a point, large is a “relative concept”.

Example 8. A million dollars is probably not a large amount of money for people like Bill Gates or George W. Bush but for “the rest of us” a million dollars is most probably a large amount of money.

Example 9. Nobody likes losing a large amount of money: Bill Gates and George W. Bush would not like to lose a billion dollars any more than “the rest of us” would like to lose a thousand dollars.
In other words, large has a meaning which is the same for everybody and it is only the cutoff point that changes from people to people.

b. However, we will need a computational definition, that is a definition that we can compute with and we will say that:

A number is large when any number of copies of that number will multiply to a number that is larger-in-size than the number itself.

**Example 10.** We will say that: $-1.1$ is large because, say, three copies of $-1.1$ multiply to $-1.331$ which is larger-in-size than $-1.1$.

c. In practice, though, whenever we will have to pick a large number to test what happens with large inputs, we will leave ourselves a “safety margin” and so:

- Computationally, we will pick $+10$ or $-10$ or any number larger-in-size than $10$.
- Graphically, we will pick numbers that are offscreen.

In other words, we will make no difference between numbers that are large and numbers that are offscreen.

2. Similarly, while the concepts of “absolutely-smaller” and “smaller in size” are both quite clear, the apparently simpler concept of “small” is in fact not at all simple to pin down. Before anything else, though, let us say that by small we will mean small in size, that is we will be ignoring the sign.

a. Up to a point, small is a “relative concept”.

**Example 11.** While, for “the rest of us” a thousand dollars is not something to be sneered at, for people like Bill Gates or George W. Bush a thousand dollars is “small”, not even a drop in a bucket.

Nevertheless, the word has a meaning which is the same for everybody and it is only the cutoff point that changes from people to people.

**Example 12.** Nobody likes to work for a small amount of money: Bill Gates and George W. Bush would not like to work a whole day for a million dollars any more than “the rest of us” would like to work a whole day for one dollar.

b. However, we will need a computational definition, that is a definition that we can compute with and we will say that:

A number is small when any number of copies of that number will multiply to a number that is smaller-in-size than the number itself.

**Example 13.** We will say that: $-0.2$ is small because, say, three copies of $-0.2$ multiply to $-0.008$ which is smaller-in-size than $-0.2$. 
c. In practice, though, whenever we will have to pick a large number to test what happens with large inputs, we will leave ourselves a “safety margin” and so:

- Computationally, we will pick $+0.1$ or $-0.1$ or any number smaller-in-size than $0.1$.
- Graphically, we will pick numbers that are near $0$.

In other words, we will make no difference between numbers that are small and numbers that are in a neighborhood of $0$.

3. There are of course numbers that are neither large nor small.

**Example 14.** $+1$ is neither large nor small because any number of copies of $+1$ will multiply to $+1$ whose size is 1, the same as the size of $+1$. Similarly, $-1$ is neither large nor small because any number of copies of $-1$ will multiply to $+1$ or $-1$ depending on whether the number of copies is even or odd but in both cases, the size of the result will be 1, the same as the size of $-1$.

In practice, the numbers that are neither large nor small are:

- computationally those numbers whose size is between $0.1$ and $10$
- graphically, those numbers that bounded numbers for the ruler

In other words, we will make no difference between numbers that are not large and number that are bounded and we will say that bounded numbers are not large and, with the exception of numbers near $0$, not small.

### 2.4 Computing With large, small And bounded

By computing with large, small and bounded, we mean something like what we mean by computing with numbers but only up to a certain point.

1. Up to a certain point, we can compare numbers that are large as well as numbers that are small according to their order of magnitude, that is, very loosely speaking, according to the number of copies that are being involved.

**Example 15.** While $10000$ and $100$ are both large, we will say that $10000$ is “larger-in-size-by-an-order-of-magnitude” than $100$ because $10000 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$ while $100 = 10 \cdot 10$

**Example 16.** While $\frac{1}{10000}$ and $\frac{1}{100}$ are both small, we will say that $\frac{1}{10000}$ is “smaller-in-size-by-an-order-of-magnitude” than $\frac{1}{100}$ because $\frac{1}{10000} = \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10}$ while $\frac{1}{100} = \frac{1}{10} \cdot \frac{1}{10}$

2. Up to a certain point, we can do the four operations with large, small and bounded:

   a. **Multiplication.** Since large, small and bounded were defined in terms of multiplication of copies, most of the following is pretty much as we would expect:
2.4. COMPUTING WITH LARGE, SMALL AND BOUNDED

**THEOREM 1 (Multiplication Of Sizes).**

\[
\begin{align*}
\text{large} \cdot \text{large} &= \text{large} & \text{large} \cdot \text{bounded} &= \text{large} & \text{large} \cdot \text{small} &= \text{any size} \\
\text{bounded} \cdot \text{large} &= \text{large} & \text{bounded} \cdot \text{bounded} &= \text{bounded} & \text{bounded} \cdot \text{small} &= \text{small} \\
\text{small} \cdot \text{large} &= \text{any size} & \text{small} \cdot \text{bounded} &= \text{small} & \text{small} \cdot \text{small} &= \text{small}
\end{align*}
\]

*large · small* and *small · large*, though, present a problem because the result could be *large, small* or *bounded* depending on how small *small* is compared to how large *large* is and we will need to look at each case separately.

**EXAMPLE 17.** The following are all instances of *large · small* but different-in-size:

\[
\begin{align*}
1000 \cdot \frac{1}{10} &= 100, & 1000 \cdot \frac{1}{1000} &= 1, & 1000 \cdot \frac{1}{100000} &= \frac{1}{100}
\end{align*}
\]

**EXAMPLE 18.** The following are all instances of *small · large* but different-in-size:

\[
\begin{align*}
\frac{1}{1000} \cdot 10000 &= 100, & \frac{1}{1000} \cdot 100 &= 1, & \frac{1}{1000} \cdot 10 &= \frac{1}{100}
\end{align*}
\]

**Division.** Since division is the same as multiplication by the reciprocal, and since *large, small* and *bounded* were defined in terms of multiplication of copies, most of the following is pretty much as we would expect:

**THEOREM 2 (Division of Sizes).**

\[
\begin{align*}
\frac{\text{large}}{\text{large}} &= \text{any size} & \frac{\text{large}}{\text{bounded}} &= \text{large} & \frac{\text{large}}{\text{small}} &= \text{large} \\
\frac{\text{bounded}}{\text{large}} &= \text{small} & \frac{\text{bounded}}{\text{bounded}} &= \text{bounded} & \frac{\text{bounded}}{\text{small}} &= \text{large} \\
\frac{\text{small}}{\text{large}} &= \text{small} & \frac{\text{small}}{\text{bounded}} &= \text{small} & \frac{\text{small}}{\text{small}} &= \text{any size}
\end{align*}
\]

*large* and *small*, though, present a problem because the result could be *large, small* or *bounded* depending on how small *small* is compared to how large *large* is and we will need to look at each case separately.

**EXAMPLE 19.** The following are all instances of *large* but different-in-size:

\[
\begin{align*}
\frac{1000}{10} &= 100, & \frac{1000}{1000} &= 1, & \frac{1000}{100000} &= \frac{1}{100}
\end{align*}
\]
**EXAMPLE 20.** The following are all instances of small but different-in-size:

\[
\frac{1}{1000} = \frac{1}{1000}, \quad 10 = \frac{10}{1}, \quad \frac{1}{1000} = 1, \quad \frac{100000}{100} = 100
\]

c. Addition and Subtraction are tricky to deal with.

**EXAMPLE 21.** The following two cases are both instances of large − large but the results are different-in-size 1000.1 − 900 = 100.1, 1000.1 − 1000 = 0.1

Fortunately, we will not have to deal with either addition or subtraction in general and, in the cases that we will deal with, later on, the result will be exactly as we would expect once we will have discussed the concept of order of magnitude to help us.

### 2.5 Infinity

A conceptual difficulty is that whether a number is bounded or offscreen depends on what quantitative ruler we happen to be using at the time.

**EXAMPLE 22.** The number 3875 is bounded relative to the following ruler

but offscreen relative to the following ruler.

So, somehow, we must “define” infinity.

1. When we go straight ahead starting from a point we will call the origin, we have the feeling that we can go as far as we want and that the longer we go, the farther away from the origin we get. But this is not really true. In fact, Magellan died in 1521 while trying to go as far as he could on this earth. His ships, though, kept on going and eventually reached home, bearing witness that there was no going around the fact that the earth is round.

So, a qualitative ruler really is just part of a Magellan circle and there is always a point, “down under the origin”, which is farthest away from the origin and, as we go past this point, we start getting closer to the origin.
In a way, then, infinity is the largest possible number for which we will use the symbol $\infty$.

But there is another aspect to infinity, namely that we cannot reach infinity just as if infinity were at $0^\circ$ Kelvin, that is at "absolute zero". Now the nearer we get to infinity, the slower we are going and so we will never reach it.

3. When we look at a given qualitative ruler as part of a Magellan circle, we see that the two stretches that are beyond the extent make up in fact a single stretch of the Magellan circle whose center is roughly $\infty$ and that we will therefore call a neighborhood of $\infty$.

NOTE. The reason the parentheses at the end of the extent face $\infty$ is because the parentheses encompass a neighborhood of $\infty$.

4. While the concept of offscreen depends on the extent we happen to choose, a number is infinite if it is offscreen no matter what extent we happen to choose.

EXAMPLE 23. Walking from Philadelphia, New York is probably offscreen. With a car, though, New York becomes onscreen. Similarly, while up until 20th July 1969 (Armstrong’s walk), the moon was offscreen, it can now be onscreen. However since no human will ever walk on Jupiter, Jupiter is near infinity.

5. Very often, we will need to distinguish the two side of $\infty$ a given number is. There are two ways to do so:

i. One way is to use the sign of the numbers:
   - The numbers to the right of the extent are on the positive side of $\infty$ and we say that the numbers are near $+\infty$
   - The numbers to the left of the extent are on the negative side of $\infty$ and we say that the numbers are near $-\infty$

EXAMPLE 24.

and therefore, from the qualitative viewpoint,
$\infty$-height input

ii. The other way is to use the side of $\infty$ the numbers are:
- The numbers that are to the left of $\infty$
- The numbers that are to the right of $\infty$

**Example 25.** The sides of $\infty$ are clear on a Magellan circle:

- $-110$ is right of $\infty$

so that

- These infinite numbers are on the right side of $\infty$
- These infinite numbers are on the left side of $\infty$

It is important, though, to realize that:
- numbers that are to the left of $\infty$ are to the right of $0$
- numbers that are to the right of $\infty$ are to the left of $0$

**Example 26.**

5. Finally, not only inputs can be near $\infty$ but outputs too can be near $\infty$. We will call $\infty$-height inputs an input such that neighboring inputs
have large outputs.

**Example 27.** In both of the following, \(-3\) is an \(\infty\)-height input:

![Graph showing inputs and outputs with a focus on \(-3\) as an \(\infty\)-height input.]

### 2.6 Localization

In the course of our qualitative investigations, we will have to deal with:

- Inputs that are near \(\infty\). We will set \(x \leftarrow \text{large}\) and compute with \(\text{large}\).
- Inputs that are near 0. We will set \(x \leftarrow \text{small}\) and compute with \(\text{small}\).

In both cases, we will compute using the **Multiplication Of Sizes Theorem** and the **Division of Sizes Theorem**.

However, we will also need to deal with

- Inputs that are near a given finite input \(x_0\) (different from 0) but we will not be able to compute with them directly.

What we will have to do is to use a technique called **localization** in which we look at the inputs as being in a neighborhood of the given input:

![Diagram showing a neighborhood of \(x_0\) and how inputs are counted from the center of the neighborhood.]

and where we temporarily "count" the inputs from the center of the neighborhood which is the given bounded input which is itself temporarily relabeled 0:
We will call the number

\[ u = x - x_0 \]

the **location relative to** \( x_0 \) of the number \( x \).

**Example 28.** Given the input +3, then the location of the number +7 relative to +3 is +4:

We will say that an input \( x \) is **near** a finite input \( x_0 \) when the **location** of \( x \) relative to \( x_0 \) is **small** and we will then use the letter \( h \) instead of the letter \( u \) to mean that it is **small**.

**Example 29.** Given the input +2, then the location of the number +2.3 relative to +2 is +0.3:

We will then be able to use the **Multiplication of Sizes Theorem** and the **Division of Sizes Theorem** to compute with the number \( h \) since \( h \) is **small**.