Chapter 3

Graphic Local Analysis

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Given a function specified by a global input-output rule, we will want to get the “local graph” near certain inputs in order:
• To get its bounded graph by joining these “local graphs”—instead of plot points,
• To define and discuss “local features”.

3.1 Local Graphs

We begin by defining what “local graphs” are.

1. Given the graph of a function, the local graph near ∞ is the part of the global graph for inputs that are near ∞.

Example 1. Given the function whose global graph is

In fact, though, we will often have to look separately at:
local graph near $+\infty$
local graph near $-\infty$
local graph near $x_0$
local feature
local code
$(\ , \ )$

- the local graph near $+\infty$, i.e. the local graph for inputs left of $\infty$,
- the local graph near $-\infty$, i.e. the local graph for inputs right of $\infty$.

**Example 2.** Local graph near $-\infty$: 

Given the graph of a function and given a finite input $x_0$, the local graph near $x_0$ is the part of the graph for the inputs that are near $x_0$.

**Example 3.** The local graph near $+3$ is

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### 3.2 Local Code

Given a function and given an input, which can be $\infty$ or a finite input $x_0$, we will want to discuss the local features, that is the features of the local graph near the given input, and so we will need a local code to represent these local features.

1. Since there is no reason for the local behavior to be the same on both sides of the given input, the local code will have to take care separately of the features left of the given input and of the features right of the given input. So, we will use a pair of parentheses to represent the neighborhood with a comma in-between to represent the given input at the center of the neighborhood: $(\ , \ )$.

2. There is however a difference when the given input is a finite input $x_0$ and when the given input is $\infty$: 

### 3.2. LOCAL CODE

- When the given input is a finite input $x_0$, we are automatically facing the center $x_0$ of the neighborhood since we are facing the screen and we have:

- When the given input is $\infty$, since we are facing the screen we are not facing the center of the neighborhood. So, to face $\infty$, we have to see the input ruler as part of a *Magellan circle* and imagine ourselves “down under” so that $+\infty$ is to our left and $-\infty$ is to our right (as opposed to when we are facing the screen):

  3. Then, given a feature and given an input, $\infty$ or $x_0$,
   i. The code we will write *left* of the comma will represent the feature of the local graph *left* of the center when *facing* the center,
   ii. The code we will write *right* of the comma will represent the feature of the local graph *right* of the center when *facing* the center.

  4. In order to describe the local behavior of a given function near a given input, finite or infinite:
   i. We *highlight* a neighborhood of the given input,
   ii. We *highlight* the local graph for that neighborhood
   iii. We *read* the features from the local graph,
   iv. We *write* within the parentheses the code for the features.
3.3 Place of a Local Graph

The first two local features we will be dealing with are the local features that, together, give the place of the local graph:

1. The height-size of a given input on a given side is the size of the outputs for inputs that are near the given input on the given side. We will code the height-size within parentheses $(\ , \ )$

   - with $\infty$ to say that the height-size is infinite
   - with $\flat$ to say that the height-size is bounded
   - with $0$ to say that the height-size is infinitesimal.

**Example 4.** Given the function $JACK$ whose local graph near $+5$ is

- the height size on the left side of $+5$ is infinite
- the height size on the right side of $+5$ is infinite

which we code as follows:

Height Size $JACK$ near $+5 = (\infty, \infty)$

**Example 5.** Given the function $JILL$ whose local graph near $+5$ is

- the height size on the left side of $+5$ is bounded
- the height size on the right side of $+5$ is infinite

which we code as follows:

Height Size $JACK$ near $+5 = (\flat, \infty)$

**Example 6.** Given the function $JANE$ whose local graph near $+5$ is

- the height size on the left side of $+5$ is infinite
- the height size on the right side of $+5$ is infinitesimal

which we code as follows:

Height Size $JANE$ near $+5 = (\infty, 0)$
2. The **height-sign** of a given input on a given side is the sign of the outputs for inputs that are near the given input on the given side.

   We will *code* the height-sign inside parentheses \((+,-)\)
   - with + to say that the local graph is *above* 0,
   - with − to say that the local graph is *below* 0.

**Example 7.** Given the function *DICK* whose local graph near +5 is

   - the height sign on the *left side* of +5 is +
   - the height sign on the *right side* of +5 is +

Which we code as follows:

   **Height Sign DICK near** +5 = \((+,+))

**Example 8.** Given the function *ZOE* whose local graph near ∞ is

   - the height sign on the *left side* of ∞ is −
   - the height sign on the *right side* of ∞ is +

Which we code as follows:

   **Height Sign ZOE near** ∞ = \((-,+))

**Example 9.** Given the function *ZINN* whose local graph near +5 is

   - the height sign on the *left side* of +5 is +
   - the height sign on the *right side* of +5 is −

Which we code as follows:

   **Height Sign ZINN near** +5 = \((+,-))

3. The **Height Sign-Size** of a local graph is the *height sign* together with the *height size*. However, we will have to keep in mind an unfortunate linguistic “peculiarity”, namely that:
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• When the code for the size is ∞, the code for the sign is written before the code for the size just as if it were a signed numbers: +∞, −∞.
• But when the code for the size is 0 or ♭, the code for the sign is written after the code for the size just as if it were an exponent: 0+, 0−, ♭+, ♭−.

EXAMPLE 10. Given the function ZACH whose local graph near ∞ is

- the height sign-size on the left side of ∞ is +large
- the height sign-size on the right side of ∞ is +small

which we code as follows:

Height Sign-Size ZACH near +5 = (+∞, 0+)

3.4 ∞-Height Inputs and 0-Height Inputs

Related to the place of the local graph, the following are notable inputs:

1. The first of this kind of notable inputs is ∞-height bounded inputs, that is bounded inputs whose nearby inputs have infinite outputs.

We will say that

\[ x_0 \text{ is a } ∞\text{-height input} \]

and to denote an ∞-height input, we will write:

\[ x_∞\text{-height} \]

with the name of the function “going without saying”.

Occasionally, we will make the following distinction:

• An even ∞-height input is an ∞-height input for which the local graph looks like one of the following:

• An odd ∞-height input is an ∞-height input for which the local graph looks like one of the following:
2. The second of this kind of notable inputs is 0-height bounded inputs, that is bounded inputs whose nearby inputs have outputs equal to 0.

\( x_0 \) is a 0-height input\(^1\) and in order to denote a 0-height input we will write

\[ x_{0\text{-height}} \]

with the name of the function “going without saying”. Graphically, this means that the local graph near \( x_{0\text{-height}} \) is near the 0-output level line and thus looks like one of the following

or like one of the following

\(^1\)Educologists will surely ask why not use zero point or critical point of order 0.
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3.5 Shape of a Local Graph

The other kind of local features of a local graph that we will discuss are the two local features that together describe the shape of the local graph. Here, though, since we are only concerned with the qualitative aspect, we will need only the sign of the features and not their size.

1. The slope sign of the local graph near \( x_0 \) is whether each side of the local graph near \( x_0 \) is
   - sloping up, that is looks more or less like \( / \) in which case we will also say that the slope is positive

   **Example 11.**

   ![Graph showing slope up examples]

   - sloping down, that is looks more or less like \( \backslash \) in which case we will also say that the slope is negative

   **Example 12.**

   ![Graph showing slope down examples]

In other words, we are not taking the size of the slope into consideration. Even though + and − are the symbols that are used traditionally\(^2\),

<table>
<thead>
<tr>
<th>We will code the slope sign within parentheses ( , )</th>
</tr>
</thead>
<tbody>
<tr>
<td>• with / to say that the local graph is sloping up (positive slope),</td>
</tr>
<tr>
<td>• with \ to say that the local graph is sloping down (negative slope),</td>
</tr>
</tbody>
</table>

**Example 13.** Given the function ABEL specified by the following graph,

\(^2\)Educologists will object that while in Rome, one ought to do as Romans do and this is indeed a close call. The use of the symbols / and \( \backslash \) is intended to help the reader: (i) describe the behavior of functions by using more transparent symbols, (ii) keep in mind what is being described by not using the same symbols for height, slope and concavity, (iii) and thus generally by not adding to the present burden whereas, later, once completely familiar with the ideas themselves and should the student pursue studies in mathematics, the notational changeover should present very little difficulty, if any.
in order to code the slope-sign of \textit{ABEL} near $+2$:

\begin{itemize}
  \item[i.] We \textit{highlight} the local graph near $+2$:
  \begin{itemize}
    \item[ii.] We \textit{read} the slope-sign from the local graph (blown up here for convenience)
    \item[iii.] We \textit{write} the code within the parentheses:
      \begin{equation*}
        \text{Slope-Sign } ABEL \text{ near } +2 = (\downarrow, \downarrow)
      \end{equation*}
  \end{itemize}
\end{itemize}

\textbf{Example 14.} Given the function \textit{CATH} specified by the following graph,

in order to \textit{code} the slope-sign of \textit{CATH} near $\infty$. 
i. We highlight the local graph near $\infty$:

![Graph Highlight]

ii. We read the slope-sign off the local graph (blown up here for convenience) keeping in mind that $+\infty$ is to the left of $\infty$ and $-\infty$ is to the right of $\infty$:

![Slope Sign]

iii. We write the code within the parentheses making sure that we are facing $\infty$:

$$\text{Slope-Sign } CATH \text{ near } \infty = (\backslash, /)$$

2. The concavity sign of the local graph near $x_0$ is whether each side of the local graph near $x_0$ is

- **bending up**, that is looks like a part of a cup like $\cup$

![Example 15]

- **bending down**, that is looks like a part of a cap like $\cap$

![Example 16]

In other words, we are not taking the size of the bending into consideration. Even though $+$ and $-$ are the symbols that are used traditionally$^3$,

$^3$Ditto.
We will code the concavity sign inside parentheses \((\ ,\ )\)

- with \(\cup\) to say that the local graph is bending up,
- with \(\cap\) to say that the local graph is bending down.

**Example 17.** Given the function \(BETH\) specified by the following graph,

in order to code the slope-sign of \(BETH\) near \(-1\):

i. We highlight the local graph near \(-1\):

ii. We read the concavity-sign from the local graph (blown up here for convenience)

iii. We write the code within the parentheses:

\[
\text{Concavity Sign } BETH \text{ near } -1 = (\cup, \cap)
\]

**Example 18.** Given the function \(DAVE\) specified by the following graph,
in order to get the slope-sign of $DAVE$ near $+1$:

i. We *highlight* the local graph near $+1$:

![Highlight](image1.png)

ii. We *read* the concavity-sign from the local graph (blown up here for convenience)

![Concavity](image2.png)

iii. We *write* the code within the parentheses:

Concavity Sign $DAVE$ near $+1 = (\cup, \cap)$

**Example 19.** Given the function $NATH$ specified by the following graph,

![Graph](image3.png)

in order to *code* the slope-sign of $NATH$ near $\infty$.

i. We *highlight* the local graph near $\infty$:

![Highlight](image4.png)
ii. We read the concavity-sign off the local graph (blown up here for convenience) keeping in mind that $+\infty$ is to the left of $\infty$ and $-\infty$ is to the right of $\infty$:

\[
\begin{array}{cccccccc}
\text{Output Ruler} & & & & & & & \text{Input Ruler} \\
\text{Screen} & & & & & & & \text{Left side of } \infty \\
\end{array}
\]

iii. We write the code within the parentheses making sure that we are facing $\infty$:

Slope-Sign \text{ NATH} \text{ near } \infty = (\cup, \cap)

3.6 Feature-Sign Change Inputs

Given a feature, a sign-change input for that feature is an input for which the sign of the feature is different on the two sides of that input.\(^4\)

1. Given a function \(f\) and an input \(x_0\), when height sign is different on the two sides of \(x_0\), that is when \(\text{Height Sign } f \text{ near } x_0 = (+, -)\) or \(\text{Height Sign } f \text{ near } x_0 = (-, +)\), we will say that \(x_0\) is a height sign change input\(^5\) and use \(x_{\text{height sign change}}\) to denote a height sign change input.

\textbf{Example 20.} Given the function \textit{JANE} whose local graph near \(-4\) is

we will say that \(-4\) is a height sign change input and use \(x_{\text{height sign change}} = -4\).

\textbf{Example 21.} Given the function \textit{KANE} whose local graph near \(+1\) is

\(^4\)Educologists will surely cringe at this terminology even though, if nothing else, it has the double merit of being \textit{systematic} and \textit{self-explanatory}.

\(^5\)Here, Educologists do not seem to feel the need for a term.
we will say that +1 is a height sign change input and use $x_{\text{height sign change}} = +1$.

2. Given a function $f$ and an input $x_0$, when slope sign is different on the two sides of $x_0$, that is when Slope-sign $f$ near $x_0 = (\nearrow, \searrow)$ or Slope-Sign $f$ near $x_0 = (\searrow, \nearrow)$, we will say that $x_0$ is a slope sign change input\(^6\) and to denote a slope sign change input we will write 

$$x_{\text{slope sign change}}$$

with the name of the function “going without saying”.

**Example 22.** Given the function $MARY$ whose local graph near $+5$ is

we will say that +5 is a slope sign change input and use $x_{\text{slope sign change}} = +5$

**Example 23.** Given the function $LARS$ whose local graph near $-4$ is

we will say that +1 is a slope sign change input and use $x_{\text{slope sign change}} = +1$

3. Given a function $f$ and an input $x_0$, when concavity sign is different from one side of $x_0$ to the other side of $x_0$, that is when Height Slope $f$ near

\(^6\)Ditto.
3.7 0-SLOPE AND 0-CONCAVITY INPUTS

\( x_0 = (\cup, \cap) \) or \((\cap, \cup)\), we will say that

\( x_0 \) is a **concavity sign change input**

and to denote a slope sign change input we will write

\( x_{\text{concavity sign change}} \)

with the name of the function, \( f \), “going without saying”.

**Example 24.** Given the function \( NATE \) whose local graph near \(-4\) is

we will say that \(+5\) is a concavity sign change input and use \( x_{\text{concavity sign change}} = +5 \)

**Example 25.** Given the function \( PETE \) whose local graph near \(-4\) is

we will say that \(+1\) is a concavity sign change input and use \( x_{\text{concavity sign change}} = +1 \)

3.7 0-Slope and 0-Concavity Inputs

A third kind of notable inputs is bounded inputs whose nearby inputs have outputs whose slope or whose concavity is near 0.

1. When it is the slope near \( x_0 \) which is near 0, we will say that

\( x_0 \) is a **0-slope input**

and to denote a 0-slope input we will write

\( x_{\text{0-slope input}} \)

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7 Here again, Educologists will deplore our lack of respect for tradition and denounce our failure to use the term “inflection point.”

8 Here educologists will want to know why not critical point of order 1 or just critical point or stationary point.
with the name of the function “going without saying”.

Thus, the local graph near \( x_0 \)-slope looks like one of the following:

\[
\begin{array}{c}
f(x_0) \\
\hline \\
x_0 \text{-slope} \\
\hline \\
Input \ Ruler \\
Output \ Ruler \\
Screen
\end{array}
\]

2. When it is the \( x_0 \)-concavity near \( x_0 \) which is near 0, we will say that \( x_0 \) is a \( x_0 \)-concavity input

and to denote a \( x_0 \)-concavity input we will write

\[
\begin{array}{c}
f(x_0) \\
\hline \\
x_0 \text{-concavity} \\
\hline \\
Input \ Ruler \\
Output \ Ruler \\
Screen
\end{array}
\]

with the name of the function “going without saying”.

Thus, the local graph near \( x_0 \)-concavity looks like one of the following:

\[
\begin{array}{c}
f(x_0) \\
\hline \\
x_0 \text{-concavity} \\
\hline \\
Input \ Ruler \\
Output \ Ruler \\
Screen
\end{array}
\]

3.8 Extremum Inputs

In many applications to the real world, one needs to compare the output of a given bounded input to the outputs of nearby inputs and we now come to the third kind of notable inputs which are extremum inputs, that is bounded inputs whose output is either absolutely larger or absolutely smaller than the output of nearby inputs.

- When the output for a bounded input \( x_0 \) is absolutely larger than the output for all nearby inputs, we will say that \( x_0 \) is a maximum input

\[
\begin{array}{c}
f(x_0) \\
\hline \\
x_0 \text{-concavity} \\
\hline \\
Input \ Ruler \\
Output \ Ruler \\
Screen
\end{array}
\]

\[\text{And finally, Educologist will wonder why not \text{critical point of order } 2?}\]
3.8. EXTREMUM INPUTS

and to denote a maximum input we will write $x_{\text{maximum}}$

with the name of the function “going without saying”.

We will say that the output for $x_{\text{maximum}}$ is a maximum output.

From the graphic viewpoint, the local graph near a maximum input is entirely below the output-level line for the maximum input.

**Example 26.**

![Diagram of maximum input and output](image)

- When the output for a bounded input $x_0$ is absolutely smaller than the output for nearby inputs, we will say that $x_0$ is a minimum input and to denote a minimum input we will write $x_{\text{minimum}}$

with the name of the function “going without saying”.

We will say that the output for $x_{\text{minimum}}$ is a minimum output.

From the graphic viewpoint, the local graph near a minimum input is entirely above the output-level line for the minimum input.

**Example 28.**
NOTE. It is most important to realize that the words “maximum” and “minimum” can be very misleading:

- When the word “maximum” is used with the word input, it does not mean that the input itself is absolutely larger than all other nearby inputs. (However, according to the above definition, when the word “maximum” is used with the word output, it does mean that the output itself is absolutely larger than the outputs for all nearby inputs.)

- When the word “minimum” is used with the word input, it does not mean that the input itself is absolutely smaller than all other nearby inputs. (However, according to the above definition, when the word “minimum” is used with the word output, it does mean that the output itself is absolutely smaller than the outputs for all nearby inputs.)