Chapter 4

From Local Graphs To Global Graph


At the end of Chapter 1, we said that, a function being specified by a global input-output rule, our primary goal in this text will be to find its qualitative bounded graph.

In the case of the functions we will investigate in this text, namely ALGEBRAIC FUNCTIONS, the local graph near $\infty$ will turn out to play a fundamental role in getting the bounded graph.

EXAMPLE 1. Even though we may not be interested in people far away from where we live, these people may have a very great importance in our lives.

So, in order to find the graph of a function being given by a global input-output rule, we will always begin by getting its local graph near $\infty$.

4.1 Smoothness

Functions can be immensely complicated and extremely hard to investigate but the functions which we will investigate in this text will all be smooth, that is none of the local features will change abruptly as the input changes.\(^1\)

\(^1\)Educologists will rightfully deplore that we thereby exclude the investigation of piecewise defined functions and this indeed was a difficult decision to make: (i) The imperative here was to have a minimal treatment by polynomial approximations.
More precisely, for a function to be smooth,

- There must be no gap, that is there must be no input for which there is no output,

**Example 2.** The function specified by the graph

![Graph](image1)

is not smooth because there is a gap since the input $-3$ has no output as indicated by the hollow dot.

**Example 3.** The function specified by the graph

![Graph](image2)

is not smooth because there is a gap since the inputs between $-2$ and $+5$ have no output. Also, while the input $+5$ has no output as indicated by the hollow dot, $-2$ does have $-17$ as an output as indicated by the solid dot.

- There must be no jump, that is no input across which the height changes abruptly,

**Example 4.** The function specified by the graph

![Graph](image3)

is not smooth because the outputs for inputs immediately left of the input $-4$ are under $+7$ while the outputs for inputs immediately right of the input $-4$ are over $+9$ so that, as the input goes across the input $-4$, the height jumps, that is changes abruptly, by $+2$. Observe, though, that there is no gap since the output for 3 is $+9$ as indicated by the solid dot.

- There must be no kink, that is no input across which the slope changes abruptly.

**Example 5.** The function specified by the graph

![Graph](image4)

(ii) But an alternative would indeed be first to investigate piecewise affine functions and, since the derivative of a piecewise affine function is a piecewise constant function, then approximate other functions by piecewise affine functions. (See [http://www.freemathtexts.org/References/EXP,%20COS%20%&%20SIN_Sp1999.pdf](http://www.freemathtexts.org/References/EXP,%20COS%20%&%20SIN_Sp1999.pdf))
4.2 \(\infty\)-HEIGHT INPUTS

\(\infty\)-HEIGHT Inputs do not prevent a function from being smooth. Of course, it looks as if there is a gap and/or a jump but drawing the graph on a Magellan screen shows that this not the case so that the existence of \(\infty\)-height inputs does not prevent a function from being smooth.

a. In the case of an even \(\infty\)-height input, the bounded graph looks as if there was just an extremum that had been truncated by not having a large enough extent but, as can be seen on a Magellan screen, it is just that the extremum output is \(\infty\).

**Example 6.** Given the function whose flat graph is a Magellan screen shows that the function is smooth:

b. In the case of an odd \(\infty\)-height input, the bounded graph looks as if the output jumped from one side of \(\infty\) to the other side of \(\infty\) but, as can again be seen on a Magellan screen, the output goes smoothly from one side of \(\infty\) to the other side of \(\infty\) just as it would with any finite output.

**Example 7.** Given the function whose flat graph is a Magellan screen shows that the function is smooth:

c. And, of course, there might be more than one \(\infty\)-height input.
CHAPTER 4. FROM LOCAL GRAPHS TO GLOBAL GRAPH

4.3 Interpolation

While, as discussed in Chapter 1, to "join smoothly" plot points makes no sense whatsoever, given two local graphs, it certainly does make sense to draw a joining graph from one local graph to the other local graph. In fact, we will say that the joining graph is a smooth interpolation if:

1. the joining graph itself is smooth,
2. the transitions between the joining graph and the given two local graphs are smooth.

EXAMPLE 8. Given the function whose flat graph is a Magellan screen shows that the function is smooth:

EXAMPLE 9. Given the function whose graph is

EXAMPLE 10. Given the function whose graph is

EXAMPLE 11. Given the function whose graph is

So, the joining graph is not a smooth interpolation.

So, the joining graph is not a smooth interpolation.

So, the joining graph is not a smooth interpolation.

So, the joining graph is not a smooth interpolation.

EXAMPLE 10. Given the function whose graph is

EXAMPLE 9. Given the function whose graph is

EXAMPLE 8. Given the function whose flat graph is

EXAMPLE 11. Given the function whose graph is
4.3. INTERPOLATION

- the *joining graph* (that joins the local graph near $-\infty$ to the local graph near $+\infty$) is smooth,
- one of the *transitions* is not smooth since it has a *jump*.

So, the *joining graph* is not a *smooth interpolation*.

**EXAMPLE 12.** Given the function whose graph is

- the *joining graph* (that joins the local graph near $-\infty$ to the local graph near $+\infty$) is smooth,
- one of the *transitions* is not smooth since it has a *jump*.

So, the *joining graph* is not a *smooth interpolation*.

2. When the joining graph is a *smooth interpolation*, the joining graph is compatible with the given local graphs in the sense that, near each *transition*, the local features of the *joining graph* is the same as the local features of the *local graph*.

**EXAMPLE 13.** Given the function whose graph is

- the local features of the local graph near $-\infty$ are $/$, $\cap$, and the local features of the joining graph near the *transition* are $/$, $\cap$.
- the local features of the local graph near $+\infty$ are $/$, $\cup$, and the local features of the joining graph near the *transition* are $/$, $\cup$.

3. *Compatible*, though, is a “weaker” requirement than *smooth* in the sense that requiring only that the transitions be compatible does not ensure that the joining graph is a *smooth interpolation*.

**EXAMPLE 14.** Given the function whose graph is
forces
offscreen graph

the transitions are compatible:
• the local features of the local graph near \(-\infty\) are \(\setminus, \cap\)
  while the features of the joining graph near the transition are \(\setminus, \cap\)
• the local features of the local graph near \(+\infty\) are \(\setminus, \cap\)
  while the features of the joining graph near the transition are \(\setminus, \cap\)

but the joining graph is not a smooth interpolation because the transitions are not smooth.

4.4 The Essential Question

As mentioned at the beginning of this chapter, a function being given by an input-output rule, we will always begin by investigating its local graph near \(\infty\) because, to a significant degree, it will be the local graph near \(\infty\) that forces the bounded graph.

1. However, the local graph near \(\infty\) need not be the only part of the global graph to “lie outside the screen” and therefore need not be the only part of the global graph likely to have an influence on the bounded graph, that is on the part of the global graph to “lie inside the screen”. As might be expected, it is all that “happens outside the screen”, in any direction, that will determine (much of) what “happens inside the screen”.

EXAMPLE 15. Dinosaurs are thought to have become extinct as a result of the impact of a large meteorite coming from outer space. Where in outer space the meteorite came from hardly mattered.

More precisely, we will call offscreen graph the part of the global graph that is outside the screen and which consists both of:
• The local graph near \(\infty\),
• The local graph(s) near \(\infty\)-height bounded input(s), if any.

EXAMPLE 16. Given the function whose bounded graph is
its offscreen graph consists of the local graph near $\infty$, the local graph near $+2$ and of the local graph near $+6$

or, on a Magellan screen,

2. However, while we always have **infinite** inputs and therefore a local graph near $\infty$, a given function may or may not have $\infty$-height **bounded** inputs so that, in the investigation of a function with the goal of getting its **global graph**, after we have obtained the local graph near $\infty$, the second step will be to get the answer to

**QUESTION 1. (Essential Question)**

- Do all **bounded inputs** have **bounded outputs** (for some extent of the output ruler)

or

- Is there one or more **bounded input** that is an $\infty$-height bounded input, that is a **bounded input** whose nearby inputs have **infinite outputs** (no matter what the extent of the output ruler)?

3. Altogether then, a function being given by its **input-output rule**, our procedure for getting its **global graph** in this text is always going to be:

i. Get its local graph near $\infty$,

ii. Get the answer to the **ESSENTIAL QUESTION** to find out if there are $\infty$-height inputs. (This will involve solving an equation.)

iii. Get the local graph near the $\infty$-height inputs, if any.

iv. Interpolate the local graph near $\infty$ and the local graphs near the $\infty$-height inputs, if any.

**EXAMPLE 17.** Given a function whose local graph near $\infty$ is
• if the answer to the essential question is that “all bounded inputs have bounded outputs”, then we can interpolate smoothly the local graph near $-\infty$ across the screen to the local graph near $+\infty$ to get the essential bounded graph:

• if the answer to the essential question is that “there is one $\infty$-height bounded input”, then before we can conclude we need to find the local graph near that $\infty$-height bounded input:
  - If we find, say, that the local graph near the $\infty$-height bounded input is
    - If we find, say, that the local graph near the $\infty$-height bounded input is
4.4. THE ESSENTIAL QUESTION

- If we find, say, that the local graph near the $\infty$-height bounded input is then the essential bounded graph is

- if the answer to the ESSENTIAL QUESTION is that “there are two $\infty$-height bounded input”, then before we can conclude we need to find the local graphs near those $\infty$-height bounded inputs.

- If we find, say, that the local graphs near the $\infty$-height bounded inputs are then the essential bounded graph is

- if the answer to the ESSENTIAL QUESTION is that “there is one $\infty$-height bounded input”, then before we can conclude we need to find the local graph near that $\infty$-height bounded input.

EXAMPLE 18. Given a function whose local graph near $\infty$ is

- if the answer to the ESSENTIAL QUESTION is that “all bounded inputs have bounded outputs”, then we can extrapolate smoothly the local graph near $-\infty$ across the screen to the local graph near $+\infty$ to get the essential bounded graph:

- if the answer to the ESSENTIAL QUESTION is that “there is one $\infty$-height bounded input”, then before we can conclude we need to find the local graph near that $\infty$-height bounded input.
- If we find, say, that the local graph near the $\infty$-height bounded input is then the essential bounded graph is

- if the answer to the Essential Question is that “there are two $\infty$-height bounded inputs”, then before we can conclude we need to find the local graphs near those $\infty$-height bounded inputs.
  - If we find, say, that the local graphs near the $\infty$-height bounded inputs are then the essential bounded graph is

- If we find, say, that the local graphs near the $\infty$-height bounded inputs are then the essential bounded graph is

- If we find, say, that the local graphs near the $\infty$-height bounded inputs are then the essential bounded graph is
4.5 The Essential Bounded Graph

So, we will be able to say, in a nutshell, that it is the offscreen graph which will force the bounded graph.

In general, though, there is no guarantee whatsoever that the bounded graph that is forced by the offscreen graph will be the actual bounded graph because there are many ways to interpolate smoothly the local graphs that make up the offscreen graph.

1. Indeed, even though the joining graphs have to be compatible with the offscreen graph, that is even though the transitions must be smooth, one does not see how the local graph near $\infty$ could force all local features in the entire bounded graph.

**Example 19.** That the distant moon has an influence on what happens on earth is clear—we only have to watch the tides on an ocean shore—but, on the other hand, the phases of the moon do not seem to have an incontrovertible influence on the growth of lettuce or on the moods of the math instructor.

a. We have already seen that, given only a local graph near $\infty$, a smooth interpolation may or may not include $\infty$-height inputs.

**Example 20.** Given the local graph near $\infty$,

![Diagram](image)

the following

![Diagram](image)

are both smooth interpolations without $\infty$-height input but the following

![Diagram](image)

are both smooth interpolations with an $\infty$-height input.
b. Similarly, given even an offscreen graph, the bounded graph of the function may or may not have bumps and hiccups and fluctuations.

**Example 21.** Given the local graph near \( \infty \),

\[
\begin{array}{c|c|c|c|c}
\text{Output} & \text{Ruler} & \text{Input} & \text{Ruler} & \text{Screen} \\
\hline
+ & \infty & – & \infty \\
\hline
\text{Local Graph near } +\infty \\
\hline
\text{Local Graph near } –\infty \\
\end{array}
\]

the following is a smooth interpolation with a bump

\[
\begin{array}{c|c|c|c|c}
\text{Output} & \text{Ruler} & \text{Input} & \text{Ruler} & \text{Screen} \\
\hline
+ & \infty & – & \infty \\
\hline
\text{Local Graph near } +\infty \\
\hline
\text{Local Graph near } –\infty \\
\end{array}
\]

is a smooth interpolation with a wave while the following is a smooth interpolation with a fluctuation.

2. Then, we will say that the essential bounded graph is the simplest possible bounded graph that is compatible with the offscreen graph, that is the local graph near \( \infty \) and the local graphs near the \( \infty \)-height inputs, if any. In other words, we can state:

**Theorem 1 (Essential Bounded Graph).** For algebraic functions, the essential bounded graph is completely determined by the offscreen graph.

**Example 22.** Given the local graph near \( \infty \),

\[
\begin{array}{c|c|c|c|c}
\text{Output} & \text{Ruler} & \text{Input} & \text{Ruler} & \text{Screen} \\
\hline
+ & \infty & – & \infty \\
\hline
\text{Local Graph near } +\infty \\
\hline
\text{Local Graph near } –\infty \\
\end{array}
\]
4.6. ESSENTIAL NOTABLE INPUTS

the following

![Graph 1]

but both of the following

![Graph 2]

are not essential bounded graphs.

is an essential bounded graph

3. The essential bounded graph will be about the best we will be able to get in this text because, with the techniques available to us in this text, bumps, hiccups and fluctuations will usually be beyond our power to detect, locate and investigate.

4.6 Essential Notable Inputs

Quite often, we will be investigating the existence or non-existence of notable inputs of a given kind.

1. Given the essential bounded graph of a function, there are two possibilities:
   - The essential bounded graph may have notable inputs of that kind. This establishes the existence—but emphatically does not give the location—of at least some notable inputs of that kind. We will say that these notable inputs are essential notable inputs. But there could always be additional notable inputs of that kind that would be part of bumps or fluctuations—and that we would therefore be unable to detect.
   - The essential bounded graph may not have any notable inputs of that kind. This, though, does not establish the non-existence of notable inputs of that kind because there could always be notable inputs of that kind that would be part of bumps or fluctuations—and that we would therefore be unable to detect.

So, on the basis of the offscreen graph, all we can do is establish:
   - the existence of notable inputs of a given kind,
   - the non-existence of essential inputs of a given kind.

This is an important distinction because it will occasionally happen that we will find that a given input is a notable input of that kind. We will then say that this notable input is a non-essential notable input. So, when we will say that a notable input is non-essential, this will just be saying that the existence of this notable input was not established on the basis of the offscreen graph.
2. The general idea in discussing on the basis of the offscreen graph the existence of notable inputs of a given kind (or the non-existence of essential notable inputs of a given kind) is that it will depend on whether the local features at the transitions are the same or the opposite but this dependence is not always quite simple and we will now look at a number of EXAMPLES:

a. There are many very simple situations in which the offscreen graph consists only of the local graph near $\infty$.

**EXAMPLE 23.** Given a function whose offscreen graph is

The concavity-signs at the transition from the local graph near $-\infty$ and at the transition to the local graph near $+\infty$ are the same, $\cap$ and $\cap$:

So, there does not have to be a bounded input near which the concavity-sign changes:

In other words, the given function has no essential concavity-sign change input.

**EXAMPLE 24.** Given the function whose offscreen graph is

The concavity-signs at the transition from the local graph near $-\infty$ and at the transition to the local graph near $+\infty$ are the opposite, $\cup$ and $\cap$:
So, there has to be a bounded input near which the concavity-sign changes:

In other words, the given function has an essential concavity-sign change input, \( x_{\text{concavity-sign change}} \), but we have no way to locate it. For all we know, we could have, say, \( x_{\text{concavity-sign change}} = -391.236 \) or \( x_{\text{concavity-sign change}} = +54.06 \) as well as anything else. All we know for sure is that there has to be a concavity-sign change input somewhere.

**Example 25.** Given the function whose offscreen graph is

The slope-signs at the transition from the local graph near \(-\infty\) to the local graph near \(+\infty\) are the opposite, (\(\/\) and \(\\)):

So, there has to be a bounded input near which the slope-sign changes:

In other words, the given function has an essential slope-sign change input, \( x_{\text{slope-sign change}} \), but we have no way to locate it. For all we know, we could have, say, \( x_{\text{slope-sign change}} = -391.236 \) as well as \( x_{\text{slope-sign change}} = +54.06 \) as well as anything else. All we know for sure is that there has to be a slope-sign change input somewhere.

**b.** Even when the offscreen graph involves local graphs near \(\infty\)-height inputs, the situation can still be fairly simple.

**Example 26.** Given the function whose offscreen graph is
The slope-signs at the transition from the local graph near \(-\infty\) and at the transition to the local graph near \(x_{\infty}^{\text{height}}\) are the same, / and \(/
\): So, there does not have to be an input between \(-\infty\) and \(x_{\infty}^{\text{height}}\) near which the slope-sign changes:

But the slope-signs at the transition from the local graph near \(x_{+}^{\text{height}}\) and at the transition to the local graph near \(+\infty\) are the opposite, \(\backslash\) and \(/\n\):

So, there has to be a bounded input between \(x_{+}^{\text{height}}\) and \(+\infty\) near which the slope-sign changes:

In other words, the given function has an essential slope-sign change input, \(x_{\text{slope-sign change}}\), but we have no way to locate it. For all we know, we could have, say, \(x_{\text{slope-sign change}} = -391.236\) or \(x_{\text{slope-sign change}} = +54.06\) as well as anything else. All we know for sure is that there has to be a slope-sign change input somewhere between \(x_{\infty}^{\text{height}}\) and \(+\infty\).

c. The situation with height-sign is different from the situation with concavity-sign and slope-sign because height-sign can depend on where the 0-output level line is while concavity-sign and slope-sign never do.

**Example 27.** Given the function whose offscreen graph is
The height-signs at the transition from
the local graph near $-\infty$ to the local
graph near $+\infty$ are the same, $+$ and
$+$. 

So, there does not have to be a bounded input near which the height-sign changes as
the bounded graph could be any one of the following essential bounded graphs.

**Example 28.** Given the function whose *offscreen graph* is

The height-signs at the transition from
the local graph near $-\infty$ to the local
graph near $+\infty$ are the *opposite*, $+$ and $-$. 

So, there has to be a bounded input
near which the height-sign changes:

In other words, the given function has an *essential* height-sign change input, $x_{\text{height-sign change}}$. 
but we have no way to locate it. For all we know, we could have, say, $x_{\text{height-sign change}} = -391.236$ as well as $x_{\text{height-sign change}} = +54.06$ as well as anything else. All we know for sure is that there has to be a height-sign change input somewhere.

d. The situation can be a bit more complicated even with concavity-sign and slope-sign.

**Example 29.** Given the function whose offscreen graph is

Because the slope-signs at the transition from the local graph near $-\infty$ and at the transition to the local graph near $+\infty$ are the opposite, $\backslash$ and $\backslash$: there has to be a bounded input near which the slope-sign changes:

But even though the concavity-signs at the transition from the local graph near $-\infty$ to the local graph near $+\infty$ are the same, $\bigcup$ and $\bigcup$:

- because the concavity-sign change signs at the transition from the local graph near $-\infty$ to the local graph near $x_{\text{slope-sign change}}$ are the opposite, $\bigcup$ and $\bigcap$. 
there has to be a bounded input between $-\infty$ and $x_{\text{slope-sign change}}^-$ near which the concavity-sign changes:

- because the concavity-sign change signs at the transition from the local graph near $x_{\text{slope-sign change}}^+$ to the local graph near $+\infty$ are the opposite, $\cap$ and $\cup$

there has to be a bounded input between $x_{\text{slope-sign change}}^-$ and $-\infty$ near which the concavity-sign changes:

Altogether we have

But while we know for sure that there has to be:
- one essential slope-sign change
- two essential concavity-sign change

we have no way to locate them. All we know for sure is that there has to be one slope-sign change input and two concavity-sign change inputs somewhere.

**Example 30.** Given the function whose offscreen graph is

there does not have to be any slope-sign change input:
EXAMPLE 31. Given the function whose offscreen graph is

the slope-signs near the transition is $\left( /, \searrow \right)$ so that there has to be a bounded input near which the slope-sign is $\left( /, \searrow \right)$ and, since the input cannot be $\infty$-height input as the offscreen graph consists of just the local graph near $\infty$, that bounded input is therefore a maximum height input:

However, we do not know where $x_{\text{maximum height}}$ is located. For all we know, we could have, say, $x_{\text{maximum height}} = -391.236$ as well as $x_{\text{maximum height}} = +54.06$ as well as anything else. What we only know for sure is that there has to be a maximum height input somewhere.

EXAMPLE 32. Given the function whose offscreen graph is

the slope-signs near the transitions with the local graph near $\infty$ (the same as in EXAMPLE 32) are $\left( /, \searrow \right)$ and so there has to be a bounded input whose slope-sign is $\left( /, \searrow \right)$ but since it is an $\infty$-height input, it is not a maximum height input:

NOTE. Here we are talking about the local features at the transitions of a given joining graph and not about the local features near a given input—note the absence of surrounding parentheses—so that, even when these transitions happen to involve the local graph near $\infty$, we are still looking at the screen and are not looking from $\infty$. 