Chapter 5

Regular Power Functions - Local Analysis

Power functions are functions that multiply or divide a given number, referred to as the coefficient, by a given number of copies of the input.

More precisely, in the global input-output rule of a power function:

- The coefficient can be any given bounded number, that is any number within the extent of the input ruler, but will usually be a finite number that is a number tick-marked on the input ruler.
- The exponent, which has to be a signed counting number, is shorthand code for what the function will do to the coefficient:
  - The sign of the exponent codes whether the coefficient is to be multiplied or divided by the copies of the input:
    - + lets the coefficient be multiplied by the copies of the input,
    - – lets the coefficient be divided by the copies of the input.

**Example 1.** \( x \xrightarrow{FLIP} FLIP(x) = (-13.44)x^{6} \)

\[
= (-13.44)x \cdot x \cdot x \cdot x \cdot x \cdot x \quad \text{6 copies of } x
\]
but
\[ x \xrightarrow{\text{FLOP}} \text{FLOP}(x) = (+8 \, 273.1)x^5 \]
\[ = +8 \, 273.1 \]
\[ \frac{x \cdot x \cdot x \cdot x \cdot x}{5 \text{ copies of } x} \]

- The size of the exponent is the number of copies of the input by which the coefficient is to be multiplied or divided. 0 means the coefficient is neither to be multiplied or divided by a copy of the input.

**Example 2.**
\[ x \xrightarrow{\text{FLIP}} \text{FLIP}(x) = (13.44)x^6 \]
\[ = (13.44) \cdot \frac{x \cdot x \cdot x \cdot x \cdot x}{6 \text{ copies of } x} \]

and
\[ x \xrightarrow{\text{FLOP}} \text{FLOP}(x) = (+8 \, 273.1)x^5 \]
\[ = +8 \, 273.1 \]
\[ \frac{x \cdot x \cdot x \cdot x \cdot x}{5 \text{ copies of } x} \]

### 5.1 Input-Output Pairs

As with any function specified by a global input-output rule, in order to get the output for a given input we must:

- **i. Read the shorthand code** of the global input-output rule,
- **ii. Write the longhand** that the shorthand code stands for,
- **iii. Write the specifying phrase** for the output by indicating which given input is to replace \( x \),
- **iv. Identify the specifying phrase** by carrying out the operations.

**Example 3.** Given the function specified by the global input-output rule
\[ x \xrightarrow{\text{FLIP}} \text{FLIP}(x) = (13.44)x^6 \]
and given the input \(-3\), we get the output as follows.

- **i. We read the shorthand code** of the global input-output rule:
  
  “The output of \text{FLIP} is obtained by multiplying \(-13.44\) by 6 copies of the input.”

- **ii. We write the longhand** that the shorthand code stands for:
  
  \[ x \xrightarrow{\text{FLIP}} \text{FLIP}(x) = (13.44)x^6 \]
  
  \[ = (-13.44) \cdot \frac{x \cdot x \cdot x \cdot x \cdot x}{6 \text{ copies of } x} \]
iii. We indicate that $x$ is to be replaced by the given input $-3$

\[
\begin{array}{c|c|c|c}
 x & \text{FLIP} \rightarrow \text{FLIP}(x) & x \leftarrow -3 \\
\hline
 x & (-13.44) \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x & x \leftarrow -3 \\
\end{array}
\]

which gives us the following specifying-phrase

\[
( -13.44 ) \cdot ( -3 ) \cdot ( -3 ) \cdot ( -3 ) \cdot ( -3 ) \cdot ( -3 ) \cdot ( -3 )
\]

iv. We identify the specifying-phrase by carrying out the multiplications:

\[
= ( -13.44 ) \cdot ( +729 ) = -9797.76
\]

Here is, typically, how the computations should look like:

\[
\begin{array}{c|c|c|c}
 x & \text{FLIP} \rightarrow \text{FLIP}(x) & x \leftarrow -3 \\
\hline
 x & (-13.44) x^6 & x \leftarrow -3 \\
\end{array}
\]

\[
= ( -13.44 ) \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \mid x \leftarrow -3 \\
= ( -13.44 ) \cdot ( -3 ) \cdot ( -3 ) \cdot ( -3 ) \cdot ( -3 ) \cdot ( -3 ) \cdot ( -3 ) \\
= ( -13.44 ) \cdot ( +729 ) \\
= -9797.76
\]

Depending on the circumstances, we can then write:

\[-3 \xrightarrow{\text{FLIP}} -9797.76\]

or

\[\text{FLIP}(-3) = -9797.76\]

or

\[(-3, -9797.76) \text{ is an input-output pair for the function } \text{FLIP}\]

**Example 4.** Given the function specified by the global input-output rule

\[
x \xrightarrow{\text{FLOP}} \text{FLOP}(x) = (+5273.1) \cdot x^{-5}
\]

and given the input $-3$, we get the output as follows:

i. We read the shorthand code of the global input-output rule:
"The output of $FLOP$ is obtained by dividing +5273.1 by 5 copies of the input."

ii. We write the longhand that the shorthand code stands for:

\[
x \xrightarrow{FLOP} FLOP(x) = (+5, 273.1)x^{-5}
\]

\[
= \frac{+5, 273.1}{x \cdot x \cdot x \cdot x} \quad \text{5 copies of } x
\]

iii. We indicate that $x$ is to be replaced by the given input $-3$

\[
x \xrightarrow{FLOP} FLOP(x) \xrightarrow{|x=-3|} \quad \frac{+5, 273.1}{x \cdot x \cdot x \cdot x} \quad \text{5 copies}
\]

which gives us the following specifying-phrase

\[
= \frac{+5, 273.1}{(-3) \cdot (-3) \cdot (-3) \cdot (-3) \cdot (-3)} \quad \text{5 copies}
\]

iv. We identify the specifying-phrase by carrying out the multiplications and the division:

\[
= \frac{+5, 273.1}{-243} = -21.7
\]

Here is, typically, how the computation should look like:

\[
x \bigg|_{x=-3} \xrightarrow{FLOP} FLOP(x) \bigg|_{x=-3} = \frac{+5, 273.1}{x \cdot x \cdot x \cdot x} \bigg|_{x=-3} = \frac{+5, 273.1}{(-3) \cdot (-3) \cdot (-3) \cdot (-3) \cdot (-3)} = \frac{-243}{-243} = -21.7
\]

Depending on the circumstances, we can then write:

\[-3 \xrightarrow{FLOP} -21.47\]

or

\[FLOP(-3) = -21.47\]

or

\[(-3, -21.47) \text{ is an input-output pair for the function } FLOP\]
5.2 Types Of Input-Output Rules

The *input-output rule* of a power function thus has a number of *features* but, since we will be mostly dealing with *qualitative* investigations, these features will not be all equally important to us.

1. The three *features* that will be important for us are:
   - **Sign exponent** which can be + or −,
   - **Parity exponent** which can be *even* or *odd* depending on whether the number of copies is *even* or *odd*.
   - **Sign coefficient** which can be + or −.

From our point of view,
- **Size coefficient** will be irrelevant because of the requirement in the definition of a power function that the *coefficient* be a *bounded* number.
- **Size exponent** will *not* be an important *feature* because the particular number of copies will not matter from our *qualitative viewpoint*.

2. Accordingly, for qualitative investigations of regular power functions, we will *normalize* global input-output rules as follows:
   i. We will set the *size* of the coefficient to 1,
   ii. We will replace the *size* of the exponent:
      - by the word *even* when the exponent is even
      - by the word *odd* when the exponent is odd

**Example 5.** The function specified by the global input-output rule

\[ x \xrightarrow{BLIP} BLIP(x) = (-16000)x^{7} \]

is a power function whose *global input-output rule* has the following *features*

- **Sign exponent** BLIP = +,
- **Parity exponent** BLIP = odd.
- **Sign coefficient** BLIP = −,

and which we normalize as

\[ x \xrightarrow{BLIP} BLIP(x) = (-1)x^{odd} \]

**Example 6.** The function specified by the global input-output rule

\[ x \xrightarrow{FLOP} FLOP(x) = (\frac{72.13}{6})^{14} \]

is a power function whose *global input-output rule* has the following *features*
CHAPTER 5. REGULAR POWER FUNCTIONS - LOCAL ANALYSIS

- Sign exponent FLOP = +,
- Parity exponent FLOP = even,
- Sign coefficient FLOP = +,

and which we normalize as

\[ x \xrightarrow{\text{FLOP}} \text{FLOP}(x) = (+1)x^{\text{even}} \]

**Example 7.** The function specified by the global input-output rule

\[ x \xrightarrow{\text{PLIP}} \text{PLIP}(x) = \left( (\frac{6836}{+6836})x^{-7} \right) \]

\[ = \frac{x \cdot \ldots \cdot x}{7 \text{ copies of } x} \]

is a power function whose global input-output rule has the following features

- Sign exponent PLIP = -,
- Parity exponent PLIP = odd.
- Sign coefficient PLIP = +,

and which we normalize as

\[ x \xrightarrow{\text{PLIP}} \text{PLIP}(x) = (+1)x^{-\text{odd}} \]

**Example 8.** The function specified by the global input-output rule

\[ x \xrightarrow{\text{MILK}} \text{MILK}(x) = (+4500)x^{-6} \]

\[ = \frac{x \cdot \ldots \cdot x}{6 \text{ copies of } x} \]

is a power function whose global input-output rule has the following features

- Sign exponent MILK = -,
- Parity exponent MILK = even.
- Sign coefficient MILK = +,

and which we normalize as

\[ x \xrightarrow{\text{MILK}} \text{MILK}(x) = (+1)x^{-\text{even}} \]

3. From the qualitative viewpoint that we will be taking, there will therefore be eight types of regular power functions:
### 5.2. TYPES OF INPUT-OUTPUT RULES

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<thead>
<tr>
<th>Sign exponent</th>
<th>Parity exponent</th>
<th>Sign coefficient</th>
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<td>+</td>
<td>(+1)x^+even</td>
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**Local Analysis Near \( \infty \)**

A power function being specified by a global input-output rule, we could get the local graph near \( \infty \) directly by:

- looking at the slope between two graph points, that is by looking at the way the height changes between two inputs

and then looking at the concavity between three graph points, that is by looking at the way the slope changes between three inputs

This, though, would involve a huge amount of numerical computations each time we would need to get the local graph of a power function near \( \infty \) and so we will find it worth investing a bit of work into making the case for two theorems that will be powerful tools:

i. A theorem that says how, in the case of power functions, to get easily the place of the local graph near \(-\infty\) from the place of the local graph near \(+\infty\).

ii. A theorem that says that, in the case of power functions, the shape of the local graph depends entirely on its place, that is on where the graph points corresponding to large inputs will be.
Altogether then, in order to find the local graph near $\infty$, we will need to compute only to get the place of the local graph near $+\infty$ and the two theorems will get us the rest.

The computations for getting the place of the local graph near $+\infty$ will be based on the following “facts” from Chapter 2 - Towards Local Analysis:

- The Rule of Signs for Multiplication

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with the consequence that:
- multiplying copies of a positive input will give a positive result whether the number of copies is even or odd,
- multiplying copies of a negative input will give:
  - a positive result if the number of copies is even
  - a negative result if the number of copies is odd

- The Definition of large, namely the fact that copies of a number that is large multiply to a result that is large.
- The fact that a bounded number
  - multiplied by a number that is large gives a number that is large,
  - divided by a number that is large gives a number that is small.

5.3 Place of Local Graphs Near $+\infty$

The place of the local graph near $+\infty$ is where the graph points for positive large inputs will be:

i. SIZE: Since the input is large and since multiplying copies of large gives a large result, the size of the output will depend on whether we multiply or divide the coefficient by that large result, that is the size of the output will depend on Sign exponent:

- When Sign exponent = +, the coefficient will be multiplied by the large result and the output will be large:
5.3. PLACE OF LOCAL GRAPHS NEAR $+\infty$

- When $\text{Sign exponent} = -$, the coefficient will be divided by the large result and the output will be small.

ii. SIGN: Since multiplying copies of a positive input will give a positive result no matter what the number of copies is, the sign of the output for positive inputs will not depend on Parity exponent and will depend only on Sign coefficient.

Altogether then:

- When $\text{Sign exponent} = +$, the place of the local graph near $+\infty$ will be, depending on Sign coefficient,

- When $\text{Sign exponent} = -$, the place of the local graph near $+\infty$ will be, depending on Sign coefficient,
In practice, though, we will deal at the same time with both the size and the sign of the inputs.

**Example 9.** Given the function specified by the global input-output rule

\[ x \xrightarrow{\text{JADE}} \text{JADE}(x) = (-83.91)x^5 \]

we want to find the place of the local graph near \(+\infty\), that is the place of the graph points for positive large inputs.

We normalize the global input output rule:

\[ x \xrightarrow{\text{JADE}} \text{JADE}(x) = (-1)x^{+\text{odd}} \]

We compute the output for inputs that are \(+\text{large}\):

\[
x \bigg|_{x\to+\text{large}} \xrightarrow{\text{JADE}} \text{JADE}(x) \bigg|_{x\to+\text{large}} = (-1)x^{+\text{odd}} \bigg|_{x\to+\text{large}}
\]

\[
= (-1)(+\text{large})^{+\text{odd}}
\]

\[
= (-1) \cdot (+\text{large}) \cdot \ldots \cdot (+\text{large})
\]

\[
\text{odd number of copies of } +\text{large}
\]

\[
= (-1) \cdot (+) \cdot \ldots \cdot (+) \cdot (\text{large}) \cdot \ldots \cdot (\text{large})
\]

\[
\text{odd number of copies of } + \cdot \text{odd number of copies of } \text{large}
\]

and since,

- by the **Rule of Signs for Multiplication**, any number of copies of \(+\) multiply to \(+\)
- by the **Definition of large**, any number of copies of \(\text{large}\) multiply to \(\text{large}\)

\[
= (-1) \cdot (+\text{large})
\]

and since a **bounded coefficient multiplied** by a **large result** gives a **large number**

\[
= -\text{large}
\]
5.3. PLACE OF LOCAL GRAPHS NEAR $+\infty$

And so we have that:  

$$JADE(+\text{large}) = -\text{large}$$

and that the place of the local graph of $JADE$ near $+\infty$ is:

**Example 10.** Given the function specified by the global input-output rule

$$x \xrightarrow{\text{NADE}} NADE(x) = (-18.56)x^{-5}$$

we want to find the local place of the local graph near $+\infty$, that is the place of the graph points for positive large inputs.

We normalize the global input output rule:

$$x \xrightarrow{\text{NADE}} NADE(x) = (-1)x^{-\text{odd}}$$

We compute the output for inputs that are $+\text{large}$:

$$x \bigg|_{x=+\text{large}} \xrightarrow{\text{NADE}} NADE(x) \bigg|_{x=+\text{large}} = (-1)x^{-\text{odd}} \bigg|_{x=+\text{large}}$$

$$= (-1)(+\text{large})^{-\text{odd}}$$

$$= (\text{+large}) \cdot \ldots \cdot (\text{+large}) = -1$$

$$\frac{\text{odd number of copies of } +\text{large}}{\text{odd number of copies of } +\text{ odd number of copies of } \text{large}}$$

and since,

- by the **Rule of Signs for Multiplication**, any number of copies of $+$ multiply to $+$
- by the **Definition of large**, any number of copies of $large$ multiply to $large$

$$= -1$$

and since a bounded coefficient divided by a large result gives a small number,

$$= -\text{small}$$
And so we have that:

\[ NADE(+\text{large}) = -small \]

and that the place of the local graph of \( DATE \) near \(+\infty\) is:

5.4 Place of Local Graphs Near \(-\infty\)

The place of the local graph near \(-\infty\) is where the graph points for negative large inputs will be:

We will proceed pretty much as we did in the previous section. For the local graph near \(-\infty\), the only difference is with the sign which, with negative inputs, depends not only on Sign coefficient but also on whether the number of copies of the negative input is even or odd, that is on Parity exponent. More precisely, multiplying copies of a negative input will give:

- a positive result for an even number of copies (Parity exponent = even)
- a negative result for an odd number of copies (Parity exponent = odd)

**Example 11.** Given the function specified by the global input-output rule

\[ x \xrightarrow{KATE} KATE(x) = (-13.14) \cdot x^{-24} \]

we want to find the place of the local graph of \( KATE \) near \(-\infty\), that is the place of the graph points for negative large inputs.

We normalize the global input output rule:

\[ x \xrightarrow{KATE} KATE(x) = (-1) \cdot x^{-\text{even}} \]
We compute the output for inputs that are \(-\text{large}\):
\[
x \mid_{x=-\text{large}} \xrightarrow{KATE} KATE(x) \mid_{x=+\text{large}} = (-1)x^\text{even} \mid_{x=-\text{large}}
\]
\[
= (-1)(-\text{large})^\text{even}
\]
\[
= \frac{(-\text{large}) \cdot \ldots \cdot (-\text{large})}{\text{even number of copies of } -\text{large}}
\]
\[
= \frac{-1}{\text{even number of copies of } -\text{large} \cdot \text{even number of copies of } \text{large}}
\]

and since,
\begin{itemize}
  \item by the Rule of Signs for Multiplication, an even number of copies of \(-\) multiply to \(+\)
  \item by the Definition of large, any number of copies of large multiply to large
\end{itemize}

\[
= -1
\]

and since a bounded coefficient divided by a result that is large gives an output that is small,

\[
= -\text{small}
\]

And so we have that: \(KATE(-\text{large}) = -\text{small}\)

and that the place of the local graph of \(KATE\) near \(-\infty\) is:

5.5 Place of Local Graphs Near \(\infty\)

For the place of the local graph near \(\infty\), we compute the place of the local graph near \(+\infty\) and then we get the place of the local graph near \(-\infty\) from the place of the local graph near \(+\infty\) by way of the following

THEOREM 1 (Local Place Near \(-\infty\)). For a regular power function, the local place near \(-\infty\) is obtained by flipping the local place near \(+\infty\) according to the parity of the exponent:
• When the exponent is even, the local place near \(-\infty\) is obtained by flipping the local graph near \(+\infty\) horizontally:

Positive exponent:

\[\text{Positive exponent:} \quad \text{Output coefficient} \quad \text{Negative coefficient} \]

Negative exponent:

\[\text{Negative exponent:} \quad \text{Positive coefficient} \quad \text{Negative coefficient} \]

• When the exponent is odd, the local place near \(-\infty\) is obtained by flipping the local graph near \(+\infty\) diagonally:

Positive exponent:

\[\text{Positive exponent:} \quad \text{Output coefficient} \quad \text{Negative coefficient} \]

Negative exponent:

\[\text{Negative exponent:} \quad \text{Positive coefficient} \quad \text{Negative coefficient} \]

The case for the Local Place Near \(-\infty\) Theorem is entirely based on the Rule of Signs for Multiplication:

• When Parity exponent = even, the result of the multiplication of the copies of the input is + so that the sign of the output is the same as the sign of the coefficient so that the place near \(-\infty\) is on the same side of \(\infty\) as the place near \(+\infty\),

• When Parity exponent = odd, the result of the multiplication of the copies of the input is − so that the sign of the output is the opposite of the sign of the coefficient so that the place near \(-\infty\) is on the opposite side of \(\infty\) from the place near \(+\infty\),

Example 12. Given the function specified by the global input-output rule

\[x \xrightarrow{DAVE} DAVE(x) = (-83.17)x^{-13}\]

we want to find the place of the local graph of DAVE near \(\infty\), that is the place of the graph points for large inputs, both positive and negative.
5.5. *PLACE OF LOCAL GRAPHS NEAR* $\infty$

We normalize the global input output rule:

$$x \overset{DAVE}{\longrightarrow} DAVE(x) = (-1)x^{-\text{odd}}$$

The first step for getting the place of the local graph near $\infty$ is to find the place of the local graph near $+\infty$ and so we compute the output for inputs that are $+\text{large}$:

$$x \bigg|_{x=+\text{large}} \overset{DAVE}{\longrightarrow} DAVE(x) \bigg|_{x=+\text{large}} = (-1)x^{-\text{odd}} \bigg|_{x=+\text{large}}$$

$$= (-1) \cdot (+\text{large})^{-\text{odd}}$$

$$= \frac{(-1 \cdot (+\text{large}) \cdot \ldots \cdot (+\text{large}))}{\text{odd number of copies of } +\text{large}}$$

$$= \frac{(-1 \cdot (+) \cdot \ldots \cdot (+) \cdot (\text{large}) \cdot \ldots \cdot (\text{large}))}{\text{odd number of copies of } + \cdot \text{odd number of copies of } \text{large}}$$

and since,

- by the **Rule of Signs for Multiplication** any number of copies of $+$ multiply to $+$
- by the **Definition of large**, any number of copies of $\text{large}$ multiply to $\text{large}$

$$= \frac{-1}{+\text{large}}$$

and since a *bounded* coefficient divided by a result that is $\text{large}$ gives an output that is $\text{small}$

$$= -\text{small}$$

And so we have that:

$$DAVE(+\text{large}) = -\text{small}$$

and that the *place* of the local graph of $DAVE$ near $+\infty$ is:

The second step, to get the place of the local graph near $\infty$, is to use the **Local Place Near $-\infty$ Theorem** to get the place of the local graph of $DAVE$ near $-\infty$:

$$DAVE(-\text{large}) = +\text{small}$$

and that the *place* of the local graph of $DAVE$ near $-\infty$ is:
Altogether then, the place of the local graph near $\infty$ is:

5.6 Shape of Local Graphs Near $\infty$

The shape of the local graph of a regular power function near $\infty$ is forced by its place.

1. More precisely, we have:

THEOREM 2 (Local Shape Near $\infty$). The local graphs near $\infty$ for regular power functions with positive exponents flatline vertically as they get nearer to $\infty$:

The local graphs near $\infty$ for regular power functions with negative exponents flatline horizontally as they get nearer to $\infty$:
2. We could of course just memorize the **Local Graph Near \( \infty \) Theorem**. However, while this would of course be *very slick*, it would also be *extremely dangerous* in that we would be totally dependent on our remembering the theorem *perfectly*, with no chance of becoming aware of an error we might have made by misremembering the theorem and, even less, of recovering from that error.

We therefore want to understand the **Local Graph Near \( \infty \) Theorem** by making the case that it is true. The advantage will be that, in case of “doubt” or “suspected error”, we will be able to examine the case for whatever we will have done.

In fact, making the case is not difficult:

i. Recall that, in looking for the place of the local graph near \( \infty \), we used the fact that:

   From the **Definition of large**, copies of *large* multiply to *large*

In reality, though, the **Definition of large** says that copies of *large* multiply to *larger* and, in fact, the larger the original is, even larger is the result.

It is precisely because we ignored that that finding the local place was easy. In order to get the local graph itself, though, we would need to use the full definition but keeping track of sizes, if not difficult, would be a lot of grudge work so we will not do it here.

ii. Fortunately, another way to look at this is that the local graph near \( \infty \) could not really be any other than what the **Local Graph Near \( \infty \) Theorem** says.

**Example 13.** Consider the following place:

Both the slope and the concavity of the local graph are *forced* because:
The slope cannot be \( \backslash \) as in because, as inputs get larger, outputs would get smaller while the Definition of large says that they have to get larger. So, the slope has to be \( \backslash \).

In other words, the only possibility is \( \backslash \) and \( \cup \):

3. In practice, to get the local graph near \( \infty \) we will proceed as follows:

   i. We will compute the place of the local graph near \( +\infty \).
   
   ii. We will invoke the Local Place Near \( -\infty \) Theorem to get the place of the local graph near \( -\infty \).
   
   iii. We will invoke the Local Graph Near \( \infty \) Theorem to get the shape of the local graph near \( \infty \).

**Example 14.** Given the function specified by the global input-output rule

\[
x \xrightarrow{KATE} KATE(x) = (-13.14) \cdot x^{+24}
\]

we want to find the local graph near \( \infty \).

We **normalize** the global input output rule:

\[
x \xrightarrow{KATE} KATE(x) = (-1) \cdot x^{+\text{even}}
\]

i. To get the local place near \( +\infty \), we compute the output for inputs that are \( +\text{large} \):

\[
x \xrightarrow{x \to +\text{large}} KATE(x) \bigg|_{x \to +\text{large}} = (-1)x^{+\text{even}} \bigg|_{x \to +\text{large}}
\]

\[
= (-1) \cdot (+\text{large})^{+\text{even}}
\]
5.6. **SHAPE OF LOCAL GRAPHS NEAR** $\infty$

\[
= (-1) \cdot (+\text{large}) \cdot \ldots \cdot (+\text{large})
\]

\[
= (-1) \cdot (+) \cdot \ldots \cdot (+) \cdot \text{large} \cdot \ldots \cdot \text{large}
\]

and since,
- by the **Rule of Signs**, any number of copies of $+$ multiply to $+$
- by the **Definition of large**, copies of $\text{large}$ multiply to $\text{large}$

\[
= (-1) \cdot (+\text{large})
\]

and by the fact that a **finite** coefficient multiplied by a result that is $\text{large}$ gives an output that is $\text{large}$

\[
= -\text{large}
\]

And so we have that:

\[
KATE(+\text{large}) = -\text{large}
\]

and that the **place** of the local graph of $KATE$ near $+\infty$ is:

ii. To get the place of the local graph near $-\infty$, we use the **Local Place Near** $-\infty$ **Theorem** which says that, since the exponent is **even**, the local place near $-\infty$ is flipped **horizontally** from the local place near $+\infty$:

iii. And then, to get the local graph of $KATE$ near $\infty$, we use the **Local Shape Near** $\infty$ **Theorem** which says that the **shape** is forced by the **place**:

**Local Analysis Near** $0$
CHAPTER 5. REGULAR POWER FUNCTIONS - LOCAL ANALYSIS

A power function being specified by a *global input-output rule*, finding from the *global input-output rule* the *place* of the local graph near 0 is essentially the same as finding the place of the local graph near $\infty$ and, here too, we will invest in making the case for the corresponding two theorems. And, here again, the computations for getting the place of the local graph near $+\infty$ will be entirely based on the following “facts” from Chapter 2 - Towards Local Analysis:

- The **Rule of Signs for Multiplication**
  
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>+</td>
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<tr>
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  with the consequence that:
  - multiplying copies of a *positive* input will result in a *positive* result no matter what the number of copies is,
  - multiplying copies of a *negative* input will give:
    - a *positive* result if the number of copies is *even*
    - a *negative* result if the number of copies is *odd*
- The **Definition of small**, namely the fact that *copies of a number that is small multiply to a result that is small*
- The fact that a *bounded* number
  - *multiplied* by a number that is *small* gives a number that is *small*
  - *divided* by a number that is *small* gives a number that is *large*

5.7 Place of Local Graphs Near $0^+$

The place of the local graph near $0^+$ is the place where the graph points for *positive small inputs* will be.

**i. Size:** Since the input is *small* and since multiplying copies of *small* gives a *small result*, the *size* of the output will depend on whether we *multiply* or *divide* the coefficient by that *small result*, that is it will depend on *Sign exponent*:

- When *Sign exponent* = $+$, the coefficient will be *multiplied* by the *small result* and the output will be *small*.
- When *Sign exponent* = $-$, the coefficient will be *divided* by the *small result* and the output will be *large*.

**ii. Sign:** Since multiplying copies of a *positive* input will result in a *positive* result no matter what the number of copies is, the sign of the output will not depend on *Parity exponent* and will depend only on *Sign coefficient*.

Altogether then:
5.7. PLACE OF LOCAL GRAPHS NEAR $0^+$

- When $\text{Sign exponent} = +$, the place of the local graph near $0^+$ will be, depending on $\text{Sign coefficient}$:

![Diagram](image1)

when $\text{Sign coefficient} = +$

- When $\text{Sign exponent} = -$, the place of the local graph near $+\infty$ will be, depending on $\text{Sign coefficient}$:

![Diagram](image2)

when $\text{Sign coefficient} = +$

In practice, though, we will deal at the same time with both the size and the sign of the inputs.

**EXAMPLE 15.** Given the function specified by the **global input-output rule**

$$x \xrightarrow{\text{KATE}} \text{KATE}(x) = (-13.14) \cdot x^{+24}$$

we want to find the place of the **local graph** near $0^+$, that is the place of the graph points for **positive small inputs**.

We **normalize** the global input output rule:

$$x \xrightarrow{\text{KATE}} \text{KATE}(x) = (-1) \cdot x^{+\text{even}}$$

We **compute** the output for inputs that are $+\text{small}$:

$$x \bigg|_{x^{+\text{small}}} \xrightarrow{\text{KATE}} \text{KATE}(x) \bigg|_{x^{+\text{small}}} = (-1)x^{+\text{even}} \bigg|_{x^{+\text{small}}}$$

$$= (-1) \cdot (+\text{small})^{+\text{even}}$$

$$= (-1) \cdot (+\text{small}) \cdot \ldots \cdot (+\text{small})$$

$\text{even number of copies of } +\text{small}$

$$= (-1) \cdot (+) \cdot \ldots \cdot (+) \cdot (\text{small}) \cdot \ldots \cdot (\text{small})$$

$\text{even number of copies of } +\text{ even number of copies of } \text{small}$
and since,

- by the Rule of Signs, copies of + always multiply to +
- by the Definition of small, copies of small multiply to small

and by the fact that a finite coefficient multiplied by a result that is small gives an output that is small

\[ (-1) \cdot (+\text{small}) = -\text{small} \]

And so we have that:

\[ KATE(+\text{small}) = -\text{small} \]

and that the place of the local graph of KATE near 0+ is:

**Example 16.** Given the function specified by the global input-output rule

\[ x \xrightarrow{\text{DATE}} DATE(x) = (-13.14) \cdot x^{-8} \]

we want to find the place of the local graph near 0+, that is the place of the graph points for positive small inputs.

We normalize the global input output rule:

\[ x \xrightarrow{\text{DATE}} DATE(x) = (-1) \cdot x^{-\text{even}} \]

We compute the output for inputs that are +small:

\[
x \mid \begin{array}{c|c}
  x \xrightarrow{\text{DATE}} DATE(x) \\
  \hline
  x:=+\text{small} \mid (-1)x^{-\text{even}} \mid x:=+\text{small} \\
  \hline
  = (-1) \cdot (+\text{small})^{-\text{even}} \\
  = \frac{-1}{(+\text{small}) \cdot \ldots \cdot (+\text{small})} \left( \frac{\text{even number of copies of } +\text{small}}{\text{even number of copies of } +} \right) \\
  = \frac{-1}{(+)} \cdot \ldots \cdot (+) \cdot (\text{small}) \cdot \ldots \cdot (\text{small}) \left( \frac{\text{even number of copies of small}}{\text{even number of copies of small}} \right)
\]

and since,

- by the Rule of Signs for Multiplication, copies of + always multiply to +
- by the Definition of small, copies of small multiply to small

\[ \frac{-1}{+\text{small}} \]
and by the fact that a finite coefficient divided by a result that is small gives an output that is large

\[ = -\text{large} \]

And so we have that:

\[ \text{DATE}(+\text{small}) = -\text{large} \]

and that the place of the local graph of \( \text{DATE} \) near \( 0^+ \) is:

5.8 Place of Local Graphs Near \( 0^- \)

The place of the local graph near \( 0^- \) is the place where the graph points for negative small inputs will be.

We will proceed pretty much as we did in the previous section. When getting the local graph near \( 0^- \), the only difference is with the sign which, with negative inputs, depends not only on Sign coefficient but also on whether the number of copies of the negative input is even or odd, that is on Parity exponent. More precisely, multiplying copies of a negative input will give:

- positive when the number of copies is even, \((\text{Parity exponent} = \text{even})\)
- negative when the number of copies is odd, \((\text{Parity exponent} = \text{odd})\)

**Example 17.** Given the function specified by the global input-output rule

\[ x \xrightarrow{\text{DATE}} \text{DATE}(x) = (-13.14) \cdot x^{-24} \]

we want to find the local graph near \( 0^- \).

We normalize the global input output rule:

\[ x \xrightarrow{\text{DATE}} \text{DATE}(x) = (-1) \cdot x^{-\text{even}} \]

We compute the output for inputs that are \(-\text{small} \):

\[
\begin{align*}
    x \bigg|_{x=-(\text{small})} \xrightarrow{\text{DATE}} \text{DATE}(x) \bigg|_{x=-(\text{small})} &= (-1)x^{-\text{even}} \bigg|_{x=-(\text{small})} \\
    &= (-1) \cdot (-\text{small})^{-\text{even}} \\
    &= -1 \\
    &= \frac{(-\text{small}) \cdot \ldots \cdot (-\text{small})}{\text{even number of copies of } -\text{small}}
\end{align*}
\]
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\[
\frac{-1}{(-) \cdot \ldots \cdot (-) \cdot \text{small} \cdot \ldots \cdot \text{small}}
\]

even number of copies of \(-\)  even number of copies of \text{small}

and since,

- by the **Rule of Signs for Multiplication**, an *even* number of copies of \(-\) multiplies to +
- by the **Definition of small**, copies of \text{small} multiply to \text{small}

\[
= \frac{-1}{+\text{small}}
\]

and by the fact that a *bounded* coefficient *divided* by a result that is \text{small} gives an output that is \text{large}

\[
= -\text{large}
\]

And so we have that:

\[
\text{DATE}(-\text{small}) = -\text{large}
\]

And that the *place* of the local graph of \text{DATE} near \(0^-\) is:

### 5.9 Place of Local Graphs Near 0

When we need the place of the local graph near \(0^-\) as second step for getting the place of the local graph near \(0\), then things are much easier because we can then get the place of the local graph near \(0^-\) from the place of the local graph near \(0^+\) with the aid of the following

**THEOREM 3 (Local Place Near 0^-).** For a regular power function, the local place near \(0^-\) is obtained by flipping the local place near \(0^+\) according to the parity of the exponent:

- When the exponent is even, the local place near \(0^-\) is obtained by flipping the local graph near \(0^+\) horizontally:
5.9. PLACE OF LOCAL GRAPHS NEAR 0

- When the exponent is odd, the local place near $0^-$ is obtained by flipping the local graph near $0^+$ diagonally:

The case for the **Local Place Near $0^-$ Theorem** is exactly the same as the case for the **Local Place Near $-\infty$ Theorem** and is entirely based on the Rule of Signs for Multiplication::

**Example 18.** Given the function specified by the global input-output rule

$$x \xrightarrow{DATE} DATE(x) = (-13.14) \cdot x^{-24}$$

we want to find the place of the local graph near 0, that is the place of the graph points for small inputs, both positive and negative.

We normalize the global input output rule:

$$x \xrightarrow{DATE} DATE(x) = (-1) \cdot x^{-\text{even}}$$

The first step for getting the place of the local graph near 0 is to find the place of the local graph near $0^+$ and so we compute the output for inputs that are $+\text{small}$:

$$x \bigg|_{x=+\text{small}} \xrightarrow{DATE} DATE(x) \bigg|_{x=+\text{small}} = (-1) x^{-\text{even}} x=+\text{small}$$

$$= (-1) \cdot (+\text{small})^{-\text{even}}$$
\[ \frac{-1}{(+ \text{small}) \cdot \ldots \cdot (+ \text{small})} \]

and since,
- by the Rule of Signs for Multiplication, copies of + always multiply to +
- by the Definition of small, copies of small multiply to small

\[ \frac{-1}{+ \text{small}} \]

and by the fact that a bounded coefficient divided by a result that is small gives an output that is large

\[ = -\text{large} \]

And so we have that:

\[ DATE(+ \text{small}) = -\text{large} \]

and that the place of the local graph of DATE near \(0^+\) is:

The second step to get the place of the local graph near 0 is to use the Local Place Near 0\(^-\) Theorem to get the place of the local graph of DATE near \(0^-\):

Altogether then, the place of the local graph near 0 is
5.10 Shape of Local Graphs Near 0

The shape of the local graph of a regular power function near 0 is forced by its place.

1. More precisely, we have:

**THEOREM 4 (Local Shape Near 0).** The local graphs near 0 for regular power functions with positive exponents flatline horizontally as they get nearer to 0:

2. As in the case of the **Local Graph Near ∞ Theorem**, one could of course just memorize the **Local Graph Near 0 Theorem** but, for the same reason, the reader should make the case for the truth of the **Local Graph Near 0 Theorem** which runs just about exactly the same as for the **Local Graph Near ∞ Theorem**.

3. In practice, to get the local graph near 0 we will proceed as we did in the case of the local graph near ∞:

i. We will compute the *place* of the local graph near $0^+$. 

ii. We will invoke the **Local Place Near 0− Theorem** to get the *place* of the local graph near $0^-$. 

iii. We will invoke the **Local Graph Near 0 Theorem** to get the local graph near 0.
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EXAMPLE 19. Given the function specified by the global input-output rule

\[ x \xrightarrow{KATE} KATE(x) = (-13.14) \cdot x^{+24} \]

we want to find the local graph near 0.
We normalize the global input output rule:

\[ x \xrightarrow{KATE} KATE(x) = (-1) \cdot x^{+even} \]

i. To get the local place near \(+\infty\), we compute the output for inputs that are \(+small\):

\[
x \bigg| \begin{array}{c|c}
  x & KATE(x) \\
\end{array} \bigg| _{x\to+small} = (-1)^{even} \cdot \underbrace{x \cdot +}_{\text{even number of copies of } +} \underbrace{\cdot \text{ small} \cdot \ldots \cdot \text{ small}}_{\text{even number of copies of } small}
\]

and since,

- by the Definition of small, copies of small multiply to small
- by the Rule of Signs, copies of + always multiply to +

\[ = (-1) \cdot (+small) \]

and by the fact that a finite coefficient multiplied by a result that is small-in-size gives an output that is small-in-size

\[ = -small \]
And so we have that:

\[ KATE(+\text{small}) = -\text{small} \]

and that the place of the local graph of \( KATE \) near \( 0^+ \) is:

ii. To get the place of the local graph near \( -\infty \), we use the Local Place Near \( -\infty \) Theorem which says that, since the exponent is even, the local place near \( -\infty \) is flipped horizontally from the local place near \( +\infty \):

iii. And then, to get the local graph of \( KATE \) near \( \infty \), we use the Local Shape Near \( \infty \) Theorem which says that the shape is forced by the place:

\[ \text{EXAMPLE 20.} \] Given the function specified by the global input-output rule

\[ x \xrightarrow{DAVE} DAVE(x) = (+83.17)x^{+13} \]

we want to find the local graph of \( DAVE \) near \( 0 \).

We normalize the global input output rule:

\[ x \xrightarrow{DAVE} DAVE(x) = (+1)x^{+\text{odd}} \]

i. To get the local place near \( 0 \), we compute the output for inputs that are \(+\text{small}\):

\[
x \bigg|_{x \to +\text{small}}^{DAVE} DAVE(x) \bigg|_{x \to +\text{small}} = (+1)x^{+\text{odd}} \bigg|_{x \to +\text{small}} = (+1) \cdot (+\text{small})^{+\text{odd}} = (+1) \cdot (+\text{small}) \cdot \ldots \cdot (+\text{small})^{\text{odd number of copies of } +\text{small}}
\]
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\[ = (+1) \cdot (\cdot \cdot \cdot (+) \cdot (small) \cdot \cdot \cdot (small)) \]

and since,
- by the Rule of Signs, copies of + always multiply to +
- by Definition of small, copies of small multiply to small

\[ = (+1) \cdot (small) \]

and, by the fact that a finite coefficient multiplied by a result that is small-in-size gives an output that is small-in-size

\[ = +small \]

And so we have that:

\[ DAVE(+small) = +small \]

and that the place of the local graph of DAVE near \( 0^+ \) is:

\( ii. \) To get the place of the local graph near \( 0^- \), we use the Local Place Near \( 0^- \) Theorem which says that, since the exponent is even, the local place near \( 0^- \) is flipped horizontally from the local place near \( 0^+ \):

\( iii. \) And then, to get the local graph of KATE near \( \infty \), we use the Local Shape Near \( \infty \) Theorem which says that the shape is forced by the place: