Add-on operator
input function
output function
add-on function

Chapter 8

Add-On Operators

Add-on operators are "super-functions" in the sense that they are to functions what functions are to numbers: given an **input function**, an add-on operator returns an **output function** in terms of an **add-on function**:

$$\left[x \xrightarrow{INPUT} INPUT(x)\right] \xrightarrow{\textbf{\textit{ADD-ON}}_{ADD-ON\,FUNCTION}} \left[x \xrightarrow{OUTPUT} OUTPUT(x)\right]$$

where

$$OUTPUT(x) = INPUT(x) + \textit{ADD-ON FUNCTION}(x)$$

8.1 Constant Add-On Functions

The simplest add-on operators are when the add-on function is a *constant function* because the output function just returns the output of the input function with the coefficient of the constant add-on function added on.

1. When the functions are specified by *tables*, things are pretty simple.

EXAMPLE 1. When we shop online, there is often a fixed add-on "Shipping Charge", say $4.25, which we represent by the *constant* add-on function:

$$x \xrightarrow{SHIPPING_{4.25}} SHIPPING_{4.25}(x) = 4.25$$

Then, for instance:

Given the input function

PRICE LIST	
Book	List Price
Math	52.45
English	47.80
History	62.75
Biology	74.50
Poetry	64.28

the operator

$\textbf{\textit{ADD-ON}}_{SHIPPING_{4.25}}$

returns the output function

NET PRICE LIST	
Book	Net Price
Math	$52.45 + 4.25$
English	$47.80 + 4.25$
History	$62.75 + 4.25$
Biology	$74.50 + 4.25$
Poetry	$64.28 + 4.25$

In other words,

PRICE LIST	
Book	List Price
Math	52.45
English	47.80
History	62.75
Biology	74.50
Poetry	64.28

$\xrightarrow{\textbf{\textit{ADD-ON}}_{SHIPPING_{4.25}}}$

NET PRICE LIST	
Book	Net Price
Math	56.70
English	52.05
History	67.00
Biology	78.75
Poetry	68.53

2. When the functions are specified by *global input-output rules*, things are still simple.

EXAMPLE 2. Given the add-on function specified by the *global input-ouput rule*

$$x \xrightarrow{\quad NANA_{-2.45}\quad} NANA_{-2.45}(x) = -2.45$$

and given the input function specified by the *global input-ouput rule*

$$x \xrightarrow{\quad INPUT\quad} INPUT(x) = +13.72x$$

then the output function will be specified by the *global input-ouput rule*

$$\begin{aligned} x \xrightarrow{\quad OUTPUT\quad} OUTPUT(x) &= INPUT(x) + NANA_{-2.45}(x) \\ &= +13.72x - 2.45 \end{aligned}$$

so that, given for instance the input number -7.4 and while

$$\begin{aligned} INPUT(-7.4) &= (+13.72) \cdot (-7.4) \\ &= -101.528 \end{aligned}$$

we have

$$\begin{aligned} OUTPUT(-7.4) &= (+13.72) \cdot (-7.4) - 2.45 \\ &= -103.978 \end{aligned}$$

8.2 Dilation Add-On Functions

We now investigate the case when the add-on function is a *dilation function* because that will very often be the case in the rest of this text.

1. When the functions are specified by *tables*, things are pretty simple.

EXAMPLE 3. When we shop local, there is a proportional add-on "Sales Tax", say 7%, which we represent by the *dilation* add-on function:

$$x \xrightarrow{SALES\,TAX_{7\%}} SALES\,TAX_{7\%}(x) = 7\% \cdot x$$

Then, for instance:

given the input function — the operator $\textbf{ADD-ON}_{SALES\,TAX_{7\%}}$ — returns the output function

PRICE LIST

Book	List Price
Math	52.45
English	47.80
History	62.75
Biology	74.50
Poetry	64.28

NET PRICE LIST

Book	Net Price
Math	$52.45 + 7\% \cdot 52.45$
English	$47.80 + 7\% \cdot 47.80$
History	$62.75 + 7\% \cdot 62.75$
Biology	$74.50 + 7\% \cdot 74.50$
Poetry	$64.28 + 7\% \cdot 64.28$

In other words:

PRICE LIST

Book	List Price
Math	52.45
English	47.80
History	62.75
Biology	74.50
Poetry	64.28

$$\xrightarrow{\textbf{ADD-ON}_{SALES\,TAX_{7\%}}}$$

NET PRICE LIST

Book	Net Price
Math	56.12
English	51.15
History	67.14
Biology	79.72
Poetry	68.78

2. When the functions are specified by *global input-output rules*, we have that, given the add-on function specified by the global input-output rule

$$x \xrightarrow{ADD-ON\ FUNCTION_{a\cdot}} ADD\text{-}ON\ FUNCTION_{a\cdot}(x) = ax$$

and given the input function specified by the global input-output rule

$$x \xrightarrow{INPUT} INPUT(x)$$

the output function is specified by the global input-output rule

$$\begin{aligned} x \xrightarrow{OUTPUT} OUTPUT(x) &= INPUT(x) + ADD\text{-}ON\ FUNCTION(x) \\ &= INPUT(x) + ax \end{aligned}$$

EXAMPLE 4. Given the add-on dilation function $SALES\,TAX_{7\%}$ specified by the *global input-ouput rule*

$$x \xrightarrow{SALES\,TAX_{7\%}} SALES\,TAX_{7\%}(x) = 7\% \cdot x$$

stack pointwise

then, given the input function $MARBLE\ LIST\ PRICE_{+4.17}$ specified by the *global input-ouput rule*

$$x \xrightarrow{MARBLE\ LIST\ PRICE_{+4.17}} MARBLE\ LIST\ PRICE_{+4.17}(x) = +4.17 \cdot x$$

then the output function $MARBLE\ NET\ PRICE_{+4.17}$will be specified by the *global input-ouput rule*

$$\begin{aligned} x \xrightarrow{MARBLE\ NET\ PRICE} MARBLE\ NET\ PRICE(x) &= +4.17 \cdot x + 7\% \cdot x \\ &= (+4.17 + 0.07) \cdot x \\ &= +4.24 \cdot x \end{aligned}$$

so that, given for instance the input number $+7.4$, while

$$\begin{aligned} MARBLE\ LIST\ PRICE_{+4.17}(+7.4) &= (+4.17) \cdot (+7.4) \\ &= +30.86 \end{aligned}$$

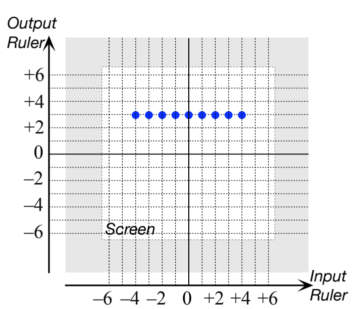
we have

$$\begin{aligned} MARBLE\ NET\ PRICE &= (+4.17) \cdot (+7.4) + (+0.07) \cdot (+7.4) \\ &= +30.86 + 0.52 \\ &= +31.38 \end{aligned}$$

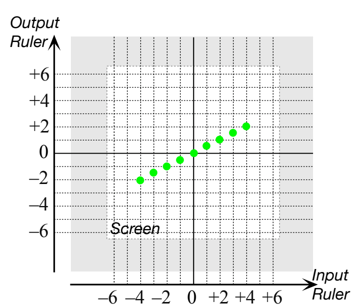
3. When the functions are specified by *quantitative plots*, things are a bit more difficult to *describe*: we must **stack pointwise** the plot of the add-on function on top of the plot of the input function. In other words, on each input-level line, we "count the output of the add-on function from the output of the input function".

EXAMPLE 5.

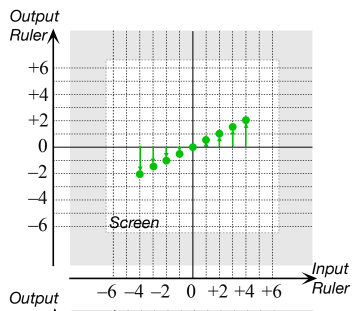
Let the input function $INPUT$ be specified by the plot:



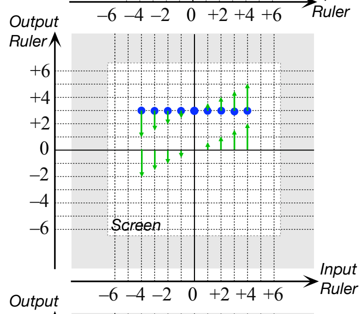
Let the add-on function $ADD\text{-}ON$ be specified by the plot:



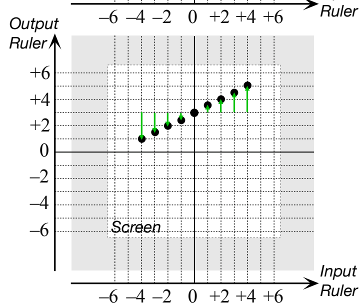
In order to get the plot for the output function $OUTPUT$ we proceed as follows: For each input, We count on the plot of $ADD\text{-}ON$ the outputs it returns



We recount these outputs on the plot of $INPUT$ starting from the outputs it returns

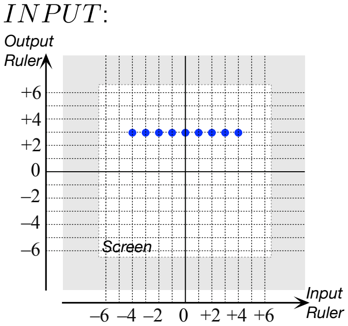


The resulting points are the plot point of $OUTPUT$

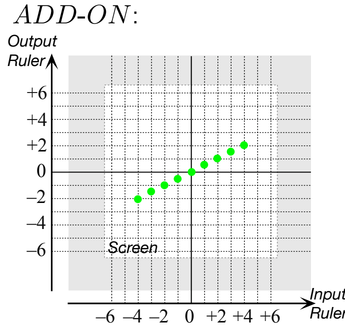


Altogether we have:

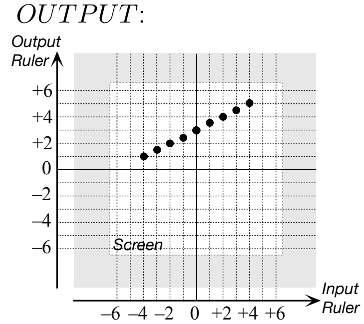
$INPUT$:



$ADD\text{-}ON$:



$OUTPUT$:



8.3 Power Add-On Functions

1. When the input function and the add-on function are specified by *global input-ouput rules*, things are fairly simple because we know how to add in algebra.

Example 6. Given the add-on function $MINT$ specified by the *global input-output rule*

$$x \xrightarrow{MINT} MINT(x) = -12.82x^{+4}$$

and given the input function TEA specified by the *global input-output rule*

$$x \xrightarrow{TEA} TEA(x) = +49.28x^{+7}$$

then the output function will be specified by the *global input-output rule*

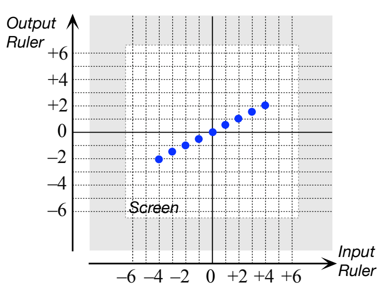
$$\begin{aligned} x \xrightarrow{\quad OUTPUT \quad} OUTPUT(x) &= TEA(x) + MINT(x) \\ &= +49.28x^{+7} - 12.82x^{+4} \end{aligned}$$

2. When the input function and the add-on function are specified by *quantitative plots*, we must *stack pointwise* the plot of the add-on function on top of the plot of the input function. In other words, on each input-level line, we "count the output of the add-on function from the output of the input function".

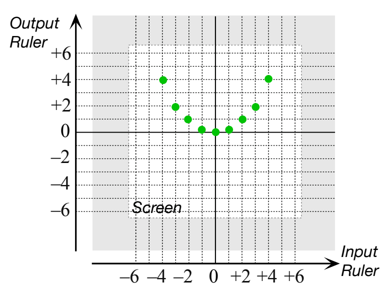
A frequent situation will be one where the input function is a *dilation function* and the add-on function is a *squaring function*.

Example 7.

Let the input function $INPUT$ be specified by the plot:



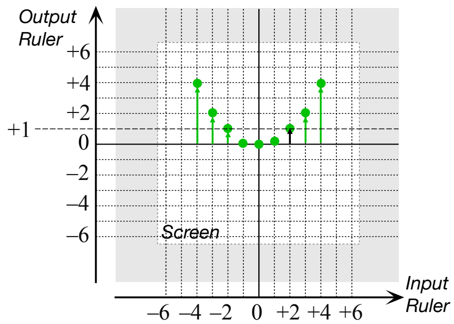
Let the add-on function $ADD\text{-}ON$ be specified by the plot:



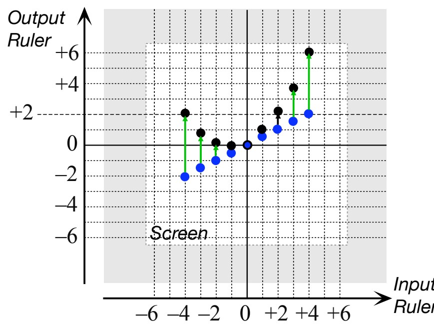
In order to get the plot for the output function $OUTPUT$ we proceed as follows:

For each input,

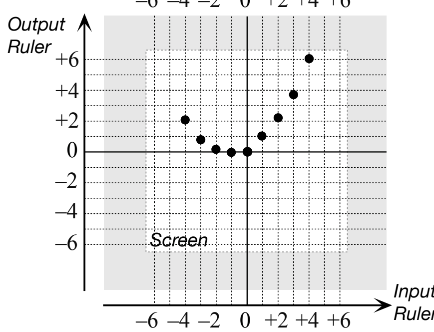
We count on the plot of *ADD-ON* the outputs it returns



We recount these outputs on the plot of *INPUT* starting from the outputs it returns

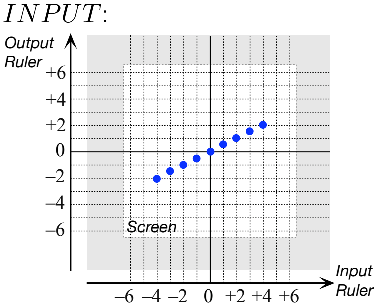


The resulting points are the plot point of *OUTPUT*

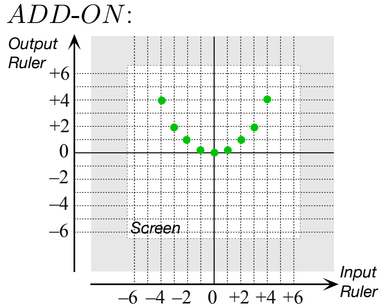


Altogether we have:

INPUT:



ADD-ON:



OUTPUT: