Chapter 3

Graphic Local Analysis

Given a function $f$ and given an input $x_0$, the local analysis of the function $f$ near the given input, be it a bounded input $x_0$ or $\infty$, is the description of the outputs returned by the function $f$ for inputs that are near the given input. In this chapter, the given function will be specified by a curve but, later on, we will derive the local graph from the given global input-output rule.

In this chapter, we will discuss feature-sign change inputs, that is inputs near which the outputs returned by the given function changes qualitatively. Locating feature-sign change inputs, though, is a global problem inasmuch as locating input(s) that meet a given requirement involves searching among all inputs.

3.1 Local Graph Near A Given Input

Given a function $f$ specified by a curve and given an input, be it a bounded input $x_0$ or $\infty$, the local graph of $f$ near that input is obtained as follows:

1. Highlight a neighborhood of that input on the input ruler,
2. Visualize the input level lines for all the inputs in the neighbor-
However, we will often look separately at:

- The local graph for inputs left of the given input when facing the center of the neighborhood).
- The local graph for inputs right of the given input when facing the center of the neighborhood.

1. When the given input is bounded, things are completely straightforward.

**Example 1.**
Given the function whose global graph is the local graph near +3 is

2. When the given input is ∞, we must take into account the fact that, since we facing the screen, we are not facing the center of the neighborhood, namely ∞.

3.

==========OK SO FAR==========
3.2. LOCAL GRAPH NEAR INFINITY

1. Given a function $f$ specified by a curve and given a bounded input $x_0$, the local graph of $f$ near $x_0$ is the part of the graph for the inputs that are near $x_0$.

3.2 Local Graph Near Infinity

1. Given a function $f$ specified by a curve, the local graph of $f$ near $\infty$ is the part of the global graph for inputs that are near $\infty$.

**Example 3.** Given the function whose global graph is

![Diagram of a function's global graph]

its local graph near $\infty$ is

![Diagram of a function's local graph near infinity]

2. Before we can deal with the sides of a local graph near $\infty$, we must discuss a difference with local graphs near a bounded input $x_0$:
   - When the given input is a bounded input $x_0$, we are automatically facing the center $x_0$ of the neighborhood since we are facing the screen:

![Diagram showing the center of the neighborhood]

   - When the given input is $\infty$, since we are facing the screen we are not facing the center of the neighborhood. So, to face $\infty$, we have to see the input ruler as part of a Magellan circle and imagine ourselves “down under” so that $+\infty$ is to our left and $-\infty$ is to our right (as opposed to when we are facing the screen):

![Diagram showing the orientation of $\infty$]

3. Thus, while facing $\infty$:
   - The local graph near $+\infty$ is the local graph for inputs left of $\infty$,
   - The local graph near $-\infty$ is the local graph for inputs right of $\infty$. 

![Diagram showing the orientation of $\infty$ and local graphs near it]
3.3 Local Code

Given a function and given an input, which can be $\infty$ or a bounded input $x_0$, we will want to describe the features of the local graph near the given input and so we will need a local code in which to write these local features.

1. Since there is no reason for the local behavior to be the same on both sides of the given input, the local code will have to take care separately of the features left of the given input and of the features right of the given input. So, we will use a pair of parentheses to represent the neighborhood with a comma in-between to represent the given input at the center of the neighborhood: $(, )$.

2. Then, given an input, be it $x_0$ or $\infty$,
   - The code left of the comma will refer to the local graph left of the center of the neighborhood (when facing the center of the neighborhood).
   - The code right of the comma will refer to the local graph right of the center of the neighborhood (when facing the center of the neighborhood).

Near a bounded input:

Near infinity:
3.4 Place of a Local Graph

The first two local features that we will be dealing with are the size and the sign of the outputs for inputs near the given input:

1. The **height-size** of a given input on a given side is the size of the outputs for inputs that are near the given input on that side.

   - We will code the height-size
     - with $\infty$ to say that the height-size is infinite,
     - with $♭$ to say that the height-size is bounded but not small,
     - with 0 to say that the height-size is small.

   **Example 5.** Given the function $JANE$ whose local graph near $+5$ is

   ![Diagram of JANE's local graph near +5]

   - the height size on the **left side** of $+5$ is **infinite**
   - the height size on the **right side** of $+5$ is **small**

   which we code as follows:

   $$\text{Height Size } JANE \text{ near } +5 = (\infty, 0)$$

2. The **height-sign** of a given input on a given side is the sign of the outputs for inputs that are near the given input on the given side.

   - We will code the height-sign
     - with $+$ to say that the local graph is **above** 0,
     - with $-$ to say that the local graph is **below** 0.

   **Example 6.** Given the function $ZOE$ whose local graph near $\infty$ is

   ![Diagram of ZOE's local graph near \infty]

   - the height sign on the **left side** of $\infty$ is $-$
   - the height sign on the **right side** of $\infty$ is $+$

   which we code as follows:

   $$\text{Height Sign } ZOE \text{ near } \infty = (-, +)$$

3. The **Height Sign-Size** of a local graph is the **height sign** together with the **height size**.

When coding the Height Sign-Size, though, we will have to keep in mind an unfortunate linguistic “peculiarity”, namely that:
• When the code for the size is \(\infty\), the code for the sign is written before the code for the size just as if it were a signed numbers: \(+\infty, -\infty\).
• But when the code for the size is 0 or \(\flat\), the code for the sign is written after the code for the size just as if it were an exponent: \(0^+, 0^-, \flat^+, \flat^-\).

**Example 7.** Given the function \(ZACH\) whose local graph near \(\infty\) is

\[
\begin{array}{c|c}
\text{Input Ruler} & \text{Output Ruler} \\
\hline
0 & -\infty \quad +\infty \\
\end{array}
\]

- the height sign-size on the left side of \(\infty\) is \(+\text{large}\)
- the height sign-size on the right side of \(\infty\) is \(-\text{small}\)

which we code as follows:

\[
\text{Height Sign-Size } ZACH \text{ near } \infty = (+\infty, 0^-)
\]

4. Together, the Height Sign and the Height Size give us the place of the local graph.

**Example 8.** If the Height Sign-Size of a function \(f\) near \(+\infty\) is \(-\infty\) then the local graph of \(f\) near \(+\infty\) is in the following place:

**Example 9.** If the Height Sign-Size of a function \(f\) near \(-\infty\) is \(0^+\) then the local graph of \(f\) near \(-\infty\) is in the following place:

**Example 10.** If the Height Sign-Size of a function \(f\) near \(-5\) is \((0^-, +\infty)\) then the local graph of \(f\) near \(-5\) is in the following place:
3.5 \(\infty\)-Height Inputs and 0-Height Inputs

Related to the height-size of the local graph, there are two kinds of notable inputs:

- A bounded input \(x_0\) is an \(\infty\)-height input if inputs that are near \(x_0\) have infinite outputs. We will use \(x_{\infty}\)-height to refer to \(\infty\)-height bounded input.
- A bounded input \(x_0\) is an 0-height input if inputs that are near \(x_0\) have small outputs. We will use \(x_{0}\)-height to refer to 0-height bounded input.

Both \(\infty\)-height inputs and 0-height inputs can be:

- even if the height-sign remains the same on both side
- odd if the height-sign changes from one side to the other.

**Example 11.** The following are \(\infty\)-height bounded inputs

**Example 12.** The following are 0-height bounded inputs

3.6 Shape of a Local Graph

We now introduce two local features of a local graph that refer to the shape of the local graph. Here, though, we will only be concerned with the quality-
slope sign
sloping up
positive slope
sloping down
negative slope
tative aspect and so we will only deal with the sign of the features and not with their size.

1. The slope sign of the local graph near $x_0$ says whether the local graph near $x_0$ is

- sloping up, that is the local graph looks more or less like / in which case we will also say that the slope is positive
- sloping down, that is the local graph looks more or less like \ in which case we will also say that the slope is negative

Even though + and − are the symbols that are used traditionally, here, for the sake of “transparency”, we will code the slope sign
- with / to say that the local graph is going up (positive slope),
- with \ to say that the local graph is going down (negative slope).

**Example 13.** Following are five local graphs together with their slope-sign:

![Local Graphs and Slope Signs](image)

**Example 14.** What is the slope-sign near $+2$ of the function $f$ specified by the curve,
We read the slope-sign from the local graph (blown up here for convenience).

Slope-Sign \textit{ABEL} near $+2 = (\cap, \cup)$

2. The \textbf{concavity sign} of the local graph near $x_0$ says whether the local graph near $x_0$ is
   - \textbf{bending up}, that is the local graph looks like part of a cup like $\cup$
   - \textbf{bending down}, that is the local graph looks like part of a cap like $\cap$

Even though $+$ and $-$ are the symbols that are used traditionally, here, for the sake of “transparency”, we will \textit{code} the \textit{concavity sign}
   - with $\cup$ to say that the local graph is \textbf{bending up (positive concavity)},
   - with $\cap$ to say that the local graph is \textbf{bending down (negative concavity)}.

\textbf{Example 15}. Following are five local graphs together with their concavity-sign:

\begin{itemize}
  \item Concave Down Up
  \item Concave Concave Down Down
  \item Concave Up Down
  \item Concave Down Up
  \item Concave Concave Down Up
\end{itemize}

Conc-sign $= (\cap, \cup)$ Conc-sign $= (\cap, \cup)$ Conc-sign $= (\cup, \cap)$ Conc-sign $= (\cap, \cup)$ Conc-sign $= (\cup, \cup)$

\textbf{Example 16}. What is the concavity-sign near $-1$ of the function $f$ specified by the curve?

concavity sign bending up cup $\cup$
concavity sign bending down cap $\cap$
sign-change input
height-sign change input

We read the concavity-sign from the local graph (blown up here for convenience)

\[
\text{Concavity-Sign } f \text{ near } -1 = (\cup, \cap)
\]

3.7 Feature-Sign Change Inputs

Given a feature, a sign-change input for that feature is an input for which the sign of the feature is different on the two sides of that input.\(^1\)

1. Given a function \( f \) and an input \( x_0 \), we will say that \( x_0 \) is a height-sign change input when height-sign is different on the two sides of \( x_0 \).

**Example 17.** Given the function \( JANE \) and given the input \(-4\),

\[-4 \text{ is a height-sign change input.}\]

**Example 18.** Given the function \( KANE \) and given the input \(+1\),

\[+1 \text{ is a height-sign change input.}\]

\(^1\)Educologists will surely cringe at this terminology even though, if nothing else, it has the double merit of being systematic and self-explanatory.
2. Given a function $f$ and an input $x_0$, we will say that $x_0$ is a **slope-sign change input** when slope-sign is different on the two sides of $x_0$.

**Example 19.** Given the function $MARY$ and given the input $+5$.

+5 is a slope-sign change input.

**Example 20.** Given the function $LARS$ and given the input $+1$.

+1 is not a slope-sign change input.

3. Given a function $f$ and an input $x_0$, we will say that $x_0$ is a **concavity-sign change input** when concavity-sign is different on the two sides of $x_0$.

**Example 21.** Given the function $NATE$ and given the input $+5$.

+5 is not a concavity-sign change input.

**Example 22.** Given the function $PETE$ whose local graph near $+1$ is
notable input
$0\text{-slope input}$
$0\text{-concavity input}$

+1 is a concavity-sign change input.

3.8 0-Slope and 0-Concavity Inputs

Related to the slope and the concavity of the local graph, the following are the third kind of notable inputs we will be looking for:

1. A bounded input $x_0$ is an 0-slope input if inputs that are near $x_0$ have small slope. We will use $x_0$-slope to refer to 0-slope inputs.

**Example 23.** The following are two examples of 0-slope bounded inputs

2. A bounded input $x_0$ is an 0-concavity input if inputs that are near $x_0$ have small concavity. We will use $x_0$-concavity to refer to 0-concavity inputs.

**Example 24.** The following are two examples of $\infty$-concavity bounded inputs
3.9. EXTREME-HEIGHT INPUTS

In many applications to the real world, one needs to compare the height of a given bounded input to the height of nearby inputs. An extreme-height input is a bounded input whose output is either absolutely larger than the height of all nearby inputs or absolutely smaller than the height of all nearby inputs.

• When the height of the extreme-height input $x_0$ is absolutely larger than the height of all nearby inputs, $x_0$ is called a maximum-height input. We will use $x_{\text{maximum-height}}$ to refer to maximum-height inputs.

From the graphic viewpoint, the local graph near a maximum-height input is entirely below the output-level line for the maximum-height input.

**Example 25.**

• When the height of the extreme-height input $x_0$ is absolutely smaller than the height of all nearby inputs, $x_0$ is called a minimum-height input. We will use $x_{\text{minimum-height}}$ to refer to minimum-height inputs.

From the graphic viewpoint, the local graph near a minimum-height input is entirely above the output-level line for the minimum-height input.

**Example 26.**
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$f(x_{\text{minimum-height}})$

Output level line for $x_{\text{minimum-height}}$

Input Ruler

Output Ruler

$X_{\text{minimum-height}}$
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