Chapter 7

Exceptional Power Functions

We now investigate the exceptional power functions, that is the power functions whose exponent is either 0 or +1.

A. Power Functions with Exponent 0

The first kind of exceptional power functions are the power functions whose exponent is 0 so that the coefficient is neither multiplied nor divided by copies of the input.

**Example 1.** Let $f$ be the power function specified by the global input-output rule

$$x \xrightarrow{f} f(x) = (+71.6)x^0$$

$$= (+71.6) \cdot \text{0 copy of } x$$

$$= +71.6$$

7.1 Constant Functions

1. Because the output of a power function whose exponent is 0 is equal to the coefficient and therefore remains the same no matter what the input is, power functions whose exponent is 0 are also called **constant functions**.
As such, constant functions are usually specified by a global input-output rule of the form

\[ x \xrightarrow{f} f(x) = a \]

where \( a \) is called the **constant coefficient**.

**EXAMPLE 2.** Let \( f \) be the function specified by the **global input-output rule**

\[ x \xrightarrow{f} f(x) = -2.45 \]

then given an input, say +61.78, we have

\[
\begin{align*}
  x &\xrightarrow{f} f(x) = -2.45 \\
  +61.78 \quad &\xrightarrow{f} f(+61.78) = -2.45
\end{align*}
\]

2. Contrary to what we did with **regular power functions**, we will not normalize constant functions. Nevertheless, the constant functions with constant coefficients +1 and -1 do have special names:

- The function \( f \) specified by the **global input-output rule**
  \[ x \xrightarrow{f} f(x) = (+1)x^0 \]
  or, more simply, by the **global input-output rule**
  \[ x \xrightarrow{f} f(x) = +1 \]
  is usually called \( UNIT_+ \) so that \( UNIT_+(x) = +1 \).

- The function \( f \) specified by the **global input-output rule**
  \[ x \xrightarrow{f} f(x) = (-1)x^0 \]
  or, more simply, by the **global input-output rule**
  \[ x \xrightarrow{f} f(x) = -1 \]
  is usually called \( UNIT_- \) so that \( UNIT_-(x) = -1 \).

In this text, though, we will not use these particular constant functions other than occasionally, as "typical" examples.

### 7.2 Global Graph Of Constant Functions

Constant functions are the first of the only three kinds of functions for which we can get the global graph **directly**.

1. Since the output of a constant function is equal to the constant coefficient no matter what the input is, we have:

**THEOREM 1 (Constant Function Global Graph).** The quantitative **global graph of the constant function**

\[ x \xrightarrow{f} f(x) = a \]
7.2. **GLOBAL GRAPH OF CONSTANT FUNCTIONS**

is the output level line \(a\).

**Example 3.** Let \(f\) be the function specified by the global input-output rule

\[ x \xrightarrow{f} f(x) = (+4)x^0 \]

We have, for instance,

\[ -3 \xrightarrow{f} f(-3) = (+4)(-3)^0 = +4 \]

\[ +5 \xrightarrow{f} f(+5) = (+4)(+5)^0 = +4 \]

etc.

2. The global graphs of *constant* functions are interesting to see in Magellan view because the Magellan view shows that, as it gets further and further away, any number dwindles down to nothing while the Mercator view gives the impression that a number remains the same no matter how far it is.

**Example 4.** Here is the global graph of the function \(UNIT_+\):

**In Mercator view:**

**In Magellan view:**

**Example 5.** Here is the global graph of the function \(UNIT_-\):

**In Mercator view:**

**In Magellan view:**
7.3 Local Graphs Of Constant Functions

In the rest of this text, though, given a constant function, we will usually need only the local graph near $\infty$ and the local graph near 0.

**Example 6.** Here are the two local graphs of the function $UNIT_+$ that we will use:

Near $\infty$:

Near 0:

**Example 7.** Here are the two local graphs of the function $UNIT_-$ that we will use:

Near $\infty$:

Near 0:

7.4 Features Of Constant Functions

What makes constant functions *exceptional* among power functions is that they lack both *local slope* and *local concavity* and have only *local height*.

But then, since for a constant function the *local height* is the same everywhere, we can talk of the **global height** of a constant function.

**Example 8.** Let $f$ be the function specified by the *global input-output rule*

\[
\begin{align*}
  x &\quad \rightarrow \quad f(x) = (-31.72)x^0 \\
               &\quad \rightarrow \quad = -31.72
\end{align*}
\]

the global height of $f$ is $-31.72$: 
7.5. **DILATION FUNCTIONS**

**B. POWER FUNCTIONS WITH EXPONENT +1**

The second kind of *exceptional* power functions are the power functions whose exponent is \[+1\] so that the coefficient is multiplied by a single copy of the input.

**Example 9.** Let \( f \) be the function specified by the *global input-output rule*

\[
x \xrightarrow{\text{BINK}} \text{BINK}(x) = (-26.18)x^1
\]

\[= (-26.18) \cdot 1 \text{ copy of } x \]

\[= (-26.18) \cdot x\]

**7.5 Dilation Functions**

1. Because the output of a power function whose exponent is \[+1\] is the coefficient multiplied by the input, power functions whose exponent is \[+1\] are also called dilation functions. As such, dilation functions are usually specified by a global input-output rule of the form

\[x \xrightarrow{f} f(x) = ax\]

where \(a\) is called the *dilation coefficient*.

**Example 10.** Let \( f \) be the function specified by the *global input-output rule*

\[x \xrightarrow{f} f(x) = -5.2\]

then, given an input, say \(-39.72\), we have

\[
x | x \leftarrow -39.72 \xrightarrow{f} f(x) | x \leftarrow -39.72 = -5.2 \cdot (-39.72) = +2038.7744
\]

**Note.** While the term “dilation function” is understood by both mathematicians and teachers, mathematicians usually prefer the term “linear function” while teachers—always a bit behind the times—still use the term “linear function” for something else that mathematicians now call “affine function” and which will be the term that will be used in Chapter 9 and 10.

2. Contrary to what we did with regular power functions—and just like what we did with constant functions, we will *not* normalize dilation functions. Nevertheless, the dilation functions with dilation coefficients \(+1\) and \(-1\) do have special names:
• The function $f$ specified by the global input-output rule
  \[ x \rightarrow f(x) = (+1)x+1 \]
or, more simply, by the global input-output rule
  \[ x \rightarrow f(x) = +x \]
is usually called \textit{IDENTITY} so that \textit{IDENTITY}(x) = +x.

• The function $f$ specified by the global input-output rule
  \[ x \rightarrow f(x) = (-1)x+1 \]
or, more simply, by the global input-output rule
  \[ x \rightarrow f(x) = -x \]
is usually called \textit{OPPOSITE} so that \textit{OPPOSITE}(x) = -x.

In this text, though, we will not use these particular dilation functions other than occasionally, as "typical" examples.

7.6 Global Graph Of Dilation Functions

\textit{Dilation functions} are the second of the three kinds of \textit{straight functions}, that is of functions whose \textit{quantitative global graph} is a \textit{straight line}. With \textit{dilation functions}, though, it is not as easy to make the case as with \textit{constant functions} because making the case requires a theorem from \textsc{Geometry} called \textbf{Thales’ Theorem}. So, here we will take for granted that the global graph of a dilation function is a \textit{straight line} and we will just focus on \textit{how} to get this straight line.

1. The key to finding where the global graph is another theorem from \textsc{Geometry}, namely that a \textit{straight line} is specified once we know two of its points. As a consequence, the quantitative global graph of a \textit{dilation function} will be specified by \textit{two} input-output pairs. There is no restriction as to what the two inputs are and the choice depends on the circumstances. Nevertheless, there are two input-input pairs that are so easy to get that they are used very often.

Given \textit{any} dilation function specified by $x \rightarrow f(x) = a \cdot x$,

- Using input 0 gives:
  \[ x \big|_{x=0} \rightarrow f(x) \big|_{x=0} = a \cdot x \big|_{x=0} \]
  \[ 0 \rightarrow f(0) = a \cdot 0 = 0 \]
  and, because any number multiplied by 0 gives 0

So, $(0, 0)$ is an input-output pair.
**Example 11.** Let $f$ be the function specified by the global input-output rule

\[ x \xrightarrow{f} f(x) = +3.7x \]

Using input $0$ gives:

\[
\begin{align*}
  x \bigg|_{x=0} & \quad \xrightarrow{f} \quad f(x) \bigg|_{x=0} = +3.7x \bigg|_{x=0} \\
  0 & \quad \xrightarrow{f} \quad f(0) = +3.7 \cdot 0
\end{align*}
\]

and, because any number multiplied by $0$ gives $0$

\[ = 0 \]

So, $(0, 0)$ is an input-output pair.

- Using input $+1$ gives:

\[
\begin{align*}
  x \bigg|_{x=+1} & \quad \xrightarrow{f} \quad f(x) \bigg|_{x=+1} = a \cdot x \bigg|_{x=+1} \\
  +1 & \quad \xrightarrow{f} \quad f(+1) = a \cdot (+1)
\end{align*}
\]

and, because any number multiplied by $+1$ gives that number

\[ = a \]

So, $(+1, a)$ is an input-output pair.

**Example 12.** Let $f$ be the function specified by the global input-output rule

\[ x \xrightarrow{f} f(x) = -2.45x \]

Using input $+1$ gives:

\[
\begin{align*}
  x \bigg|_{x=+1} & \quad \xrightarrow{f} \quad f(x) \bigg|_{x=+1} = -2.45 \bigg|_{x=+1} \\
  +1 & \quad \xrightarrow{f} \quad f(+1) = -2.45 \cdot (+1)
\end{align*}
\]

and, because any number multiplied by $+1$ gives that number

\[ = -2.45 \]

So, $(1, -2.45)$ is an input-output pair.

In other words, we have the following

**Theorem 2 (Dilation Function Global Graph).** The quantitative global graph of the dilation function

\[ x \xrightarrow{f} f(x) = ax \]

is the straight line that goes through the two input-output pairs:

$(0, 0)$ and $(+1, a)$
EXAMPLE 13. Let $f$ be the function specified by the global input-output rule

\[ x \xrightarrow{f} f(x) = -2.5x \]

in order to get the global graph,

- Using input 0 gives:

\[
\begin{align*}
  x & \bigg|_{x=0} \xrightarrow{f} f(x) \bigg|_{x=0} = -2.5 \bigg|_{x=0} \\
  &= -2.5 \cdot (0) \\
  &= 0
\end{align*}
\]

- Using input +1 gives:

\[
\begin{align*}
  x & \bigg|_{x=+1} \xrightarrow{f} f(x) \bigg|_{x=+1} = -2.5 \bigg|_{x=0} \\
  &= -2.5 \cdot (+1) \\
  &= -2.5
\end{align*}
\]

So we plot the input-output pairs $(0, 0)$ and $+1, -2.5$

And then we draw the straight line that goes through the two plot points.

2. The global graph of dilation functions are also interesting to see in Magellan view.

EXAMPLE 14. Here is the global graph of the function $IDENTITY$:
7.7 Local Graphs Of Dilation Functions

In the rest of this text, though, given a dilation function, we will usually need only the local graph near $\infty$ and the local graph near 0.

**Example 16.** Here are the two local graphs of the $\text{IDENTITY}$ function that we will use:

Near $\infty$:

Near 0:

**Example 17.** Here are the two local graphs of the $\text{OPPOSITE}$ function that we will use:
7.8 Features Of Dilation Functions

What makes dilation functions exceptional among power functions is that they lack local concavity and have only local height and local slope.

But then, since for a dilation function the local slope is the same everywhere, we can talk of the global slope of a dilation function.

**Example 18.** Let \( f \) be the function specified by the global input-output rule

\[
x \xrightarrow{f} f(x) = (0.5)x^{+1}
\]

\[
= +0.5x
\]

the global slope of \( f \) is +0.5:
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