operator
input function
output function
add-on operators
add-on function
row-wise

Chapter 8

Add-On Operators

Operators are "super-functions" in the sense that they do with functions what functions do with numbers. More precisely, given an **input function**, an operator returns an **output function**. So, if $INPUT$ is the input function and if $OUTPUT$ if the function that the operator outputs, then we have

$$\underbrace{\left[x \xrightarrow{INPUT} INPUT(x)\right]}_{\text{Input function}} \quad \overset{\textbf{OPERATOR}}{\Longrightarrow} \quad \underbrace{\left[x \xrightarrow{OUTPUT} OUTPUT(x)\right]}_{\text{Output function}}$$

In this text, though, we will use only **add-on operators**, that is operators that add to $INPUT(x)$, the output of the input function $INPUT$, the output of a given function, referred to as **add-on function**. In other words, if $ADD\text{-}ON$ is the given add-on function, then, for each input x, we have:

$$OUTPUT(x) = INPUT(x) + ADD\text{-}ON\,(x)$$

8.1 Functions Specified by Input-Output Tables

Even though later we will not deal with functions specified by *input-output tables*, here we will do so in order to illustrate the way add-on operators work because, when the functions are specified by *input-output tables*, things are pretty simple as we add-on **row-wise**.

1. The simplest add-on operators are those where the add-on function is a *constant function* so that, for any input, the output function returns the output of the input function plus the *constant output* of the add-on function.

EXAMPLE 1. When we shop online, a fixed *shipping charge* is usually added-on to the *list price* to get the *pay price*. For instance, in the case of textbooks,

Given the function the *add-on operator* returns the function

$LIST\ PRICES$	
x	$LIST(x)$
Math	52.45
English	47.80
History	62.75
Biology	74.50
Poetry	64.25

Input function

$SHIPPING$

x	$SHIP(x)$
Math	3.45
English	3.45
History	3.45
Biology	3.45
Poetry	3.45

$PAY\ PRICES$	
x	$PAY(x)$
Math	55.90
English	51.25
History	66.20
Biology	77.95
Poetry	67.70

Output function

2. The next simplest add-on operators are those where the add-on function is a *dilation function* so that, for each input, the output function returns the output of the input function plus the dilation coefficient of the add-on function multiplied by the output of the input function.

EXAMPLE 2. When we shop local, a *sales tax* is usually added-on to the *list price* to get the *pay price*. For instance, in the case of textbooks,

Given the function the *add-on operator*: returns the function:

$LIST\ PRICES$	
x	$LIST(x)$
Math	52.45
English	47.80
History	62.75
Biology	74.50
Poetry	64.25

Input function

$SALES\ TAX$

x	$TAX(x)$
Math	$7\% \cdot 52.45$
English	$7\% \cdot 47.80$
History	$7\% \cdot 62.75$
Biology	$7\% \cdot 74.50$
Poetry	$7\% \cdot 64.25$

$PAY\ PRICES$	
x	$PAY(x)$
Math	56.12
English	51.15
History	67.14
Biology	79.72
Poetry	68.75

Output function

3. The add-on function, though, need not be as simple as in the above two cases.

EXAMPLE 3. Depending on how popular a product is, a *premium* or a *discount* may be added-on to the *list price* to get the *pay price*. For instance, in the case of textbooks,

raise/lower
tilt

Given the function: the *add-on operator*: returns the function:

$LIST\ PRICES$	
x	$LIST(x)$
Math	52.45
English	47.80
History	62.75
Biology	74.50
Poetry	64.25

Input function

$PREM/DISC$	
x	$PR/DS(x)$
Math	−6.45
English	0.00
History	+2.10
Biology	0.00
Poetry	+5.05

$PAY\ PRICES$	
x	$PAY(x)$
Math	46.00
English	47.80
History	64.85
Biology	74.50
Poetry	69.30

Output function

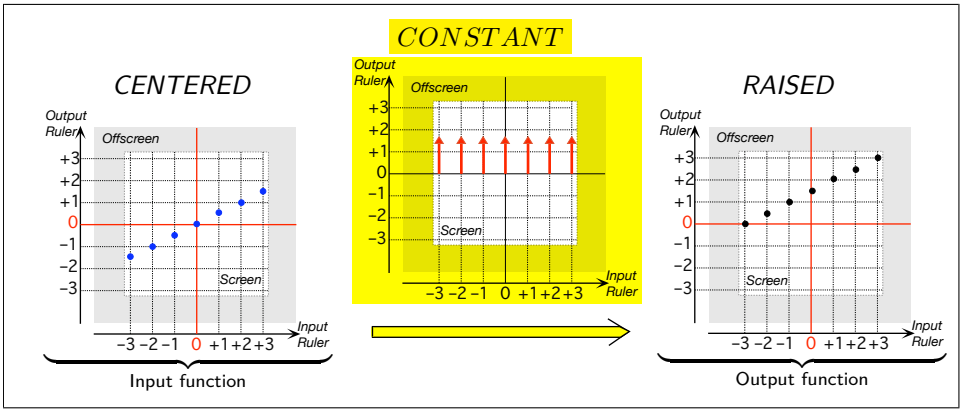
8.2 Functions Specified by Plots

Again, even though we will not deal later with functions specified by *plots*, here we will do so in order to illustrate the way add-on operators work when, as will be the case later on, functions are specified by *graphs*.

1. A frequent case will be when the add-on operator is specified by a *constant function*. The effect of such an add-on operator is to **raise/lower** the input function.

EXAMPLE 4.

Given the function: the *add-on operator*: returns the function:

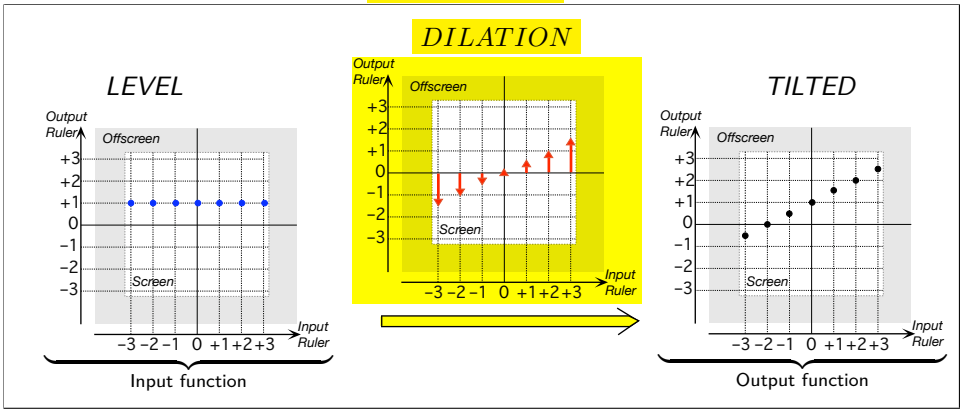


2. Another frequent case will be when the add-on operator is specified by a *dilation function*. The effect of such an add-on operator is to **tilt** the input function.

bend

EXAMPLE 5.

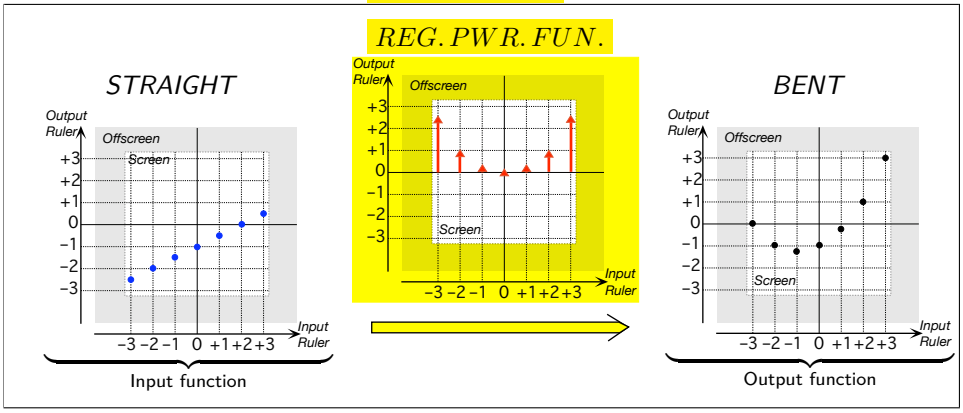
Given the function: the *add-on operator*: returns the function:



3. Finally, yet another frequent case will be when the add-on operator is specified by a *regular power function*. The effect of such an add-on operator is to **bend** the input function.

EXAMPLE 6.

Given the function: the *add-on operator*: returns the function:



8.3 Functions Specified by Input-Output Rules

=========OK SO FAR=========

When the functions are specified by *global input-output rules*, things are

still fairly simple:

Given the *input function* specified by the *global input-ouput rule*

$$x \xrightarrow{INPUT} INPUT(x) = \text{phrase specifying the output of } INPUT$$

the *add-on operator* specified by the *add-on function* specified by the *global input-ouput rule*

$$x \xrightarrow{ADD\text{-}ON} ADD\text{-}ON(x) = \text{phrase specifying the output of } ADD\text{-}ON$$

returns the *output function* specified by the *global input-ouput rule*

$$x \xrightarrow{OUTPUT} OUTPUT(x) = INPUT(x) + ADD\text{-}ON(x)$$
$$= \text{phrase specifying the output of } INPUT$$
$$+ \text{phrase specifying the output of } ADD\text{-}ON$$

1. When the add-on function f is a *constant function* specified by the global input-output rule

$$x \xrightarrow{f} f(x) = a$$

where a is the *constant coefficient*, the $OUTPUT$ function that the add-on operator returns for the $INPUT$ function is the function specified by the global input-output rule:

$$x \xrightarrow{OUTPUT} OUTPUT(x) = INPUT(x) + a$$

2. When the add-on function f is a *dilation function* specified by the global input-output rule

$$x \xrightarrow{f} f(x) = ax$$

where a is the *dilation coefficient*, the $OUTPUT$ function that the add-on operator returns for the $INPUT$ function is the function specified by the global input-output rule:

$$x \xrightarrow{OUTPUT} OUTPUT(x) = INPUT(x) + ax$$

1.

2. EXAMPLE 7. Given the (constant) add-on function specified by the *global input-ouput rule*

$$x \xrightarrow{ADD\text{-}ON_{-2.45}} ADD\text{-}ON_{-2.45}(x) = -2.45$$

then, given the *input function* specified by the *global input-ouput rule*

$$x \xrightarrow{INPUT} INPUT(x) = +13.72x$$

the output function will be specified by the *global input-ouput rule*

$$\begin{aligned} x \xrightarrow{OUTPUT} OUTPUT(x) &= INPUT(x) + ADD\text{-}ON_{-2.45}(x) \\ &= +13.72x - 2.45 \end{aligned}$$

so that given, say, the input number -7.4 we have

$$\begin{aligned} INPUT(-7.4) &= (+13.72) \cdot (-7.4) \\ &= -101.528 \end{aligned}$$

and

$$\begin{aligned} OUTPUT(-7.4) &= (+13.72) \cdot (-7.4) - 2.45 \\ &= -103.978 \end{aligned}$$

8.4 Dilation Add-On Functions

We now investigate the case when the add-on function is a *dilation function* because that will very often be the case in the rest of this text.

1. When the functions are specified by *tables*, things are pretty simple.

2. When the functions are specified by *global input-output rules*, we have that, given the add-on function specified by the global input-output rule

$$x \xrightarrow{ADD-ON\ FUNCTION_{a\cdot}} ADD\text{-}ON\ FUNCTION_{a\cdot}(x) = ax$$

and given the input function specified by the global input-output rule

$$x \xrightarrow{INPUT} INPUT(x)$$

the output function is specified by the global input-output rule

$$\begin{aligned} x \xrightarrow{OUTPUT} OUTPUT(x) &= INPUT(x) + ADD\text{-}ON\ FUNCTION(x) \\ &= INPUT(x) + ax \end{aligned}$$

EXAMPLE 8. Given the add-on dilation function $SALES\ TAX_{7\%}$ specified by the *global input-ouput rule*

$$x \xrightarrow{SALES\ TAX_{7\%}} SALES\ TAX_{7\%}(x) = 7\% \cdot x$$

then, given the input function $MARBLE\ LIST\ PRICE_{+4.17}$ specified by the *global input-ouput rule*

$$x \xrightarrow{MARBLE\ LIST\ PRICE_{+4.17}} MARBLE\ LIST\ PRICE_{+4.17}(x) = +4.17 \cdot x$$

then the output function $MARBLE\ NET\ PRICE_{+4.17}$will be specified by the *global input-ouput rule*

$$\begin{aligned} x \xrightarrow{MARBLE\ NET\ PRICE} MARBLE\ NET\ PRICE(x) &= +4.17 \cdot x + 7\% \cdot x \\ &= (+4.17 + 0.07) \cdot x \\ &= +4.24 \cdot x \end{aligned}$$

so that, given for instance the input number $+7.4$, while

$$\begin{aligned} MARBLE\ LIST\ PRICE_{+4.17}(+7.4) &= (+4.17) \cdot (+7.4) \\ &= +30.86 \end{aligned}$$

we have

$$\begin{aligned} MARBLE\ NET\ PRICE &= (+4.17) \cdot (+7.4) + (+0.07) \cdot (+7.4) \\ &= +30.86 + 0.52 \\ &= +31.38 \end{aligned}$$

3.

8.5 Power Add-On Functions

1. When the input function and the add-on function are specified by *global input-ouput rules*, things are fairly simple because we know how to add in algebra.

EXAMPLE 9. Given the add-on function $MINT$ specified by the *global input-output rule*

$$x \xrightarrow{MINT} MINT(x) = -12.82x^{+4}$$

and given the input function TEA specified by the *global input-output rule*

$$x \xrightarrow{TEA} TEA(x) = +49.28x^{+7}$$

then the output function will be specified by the *global input-output rule*

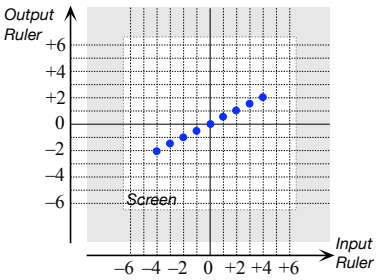
$$\begin{aligned} x \xrightarrow{OUTPUT} OUTPUT(x) &= TEA(x) + MINT(x) \\ &= +49.28x^{+7} - 12.82x^{+4} \end{aligned}$$

2. When the input function and the add-on function are specified by *quantitative plots*, we must *stack pointwise* the plot of the add-on function on top of the plot of the input function. In other words, on each input-level line, we "count the output of the add-on function from the output of the input function".

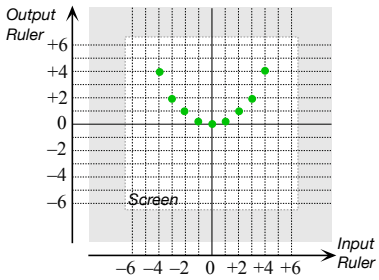
A frequent situation will be one where the input function is a *dilation function* and the add-on function is a *squaring function*.

EXAMPLE 10.

Let the input function $INPUT$ be specified by the plot:



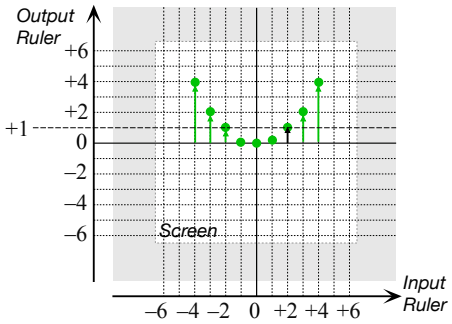
Let the add-on function $ADD\text{-}ON$ be specified by the plot:



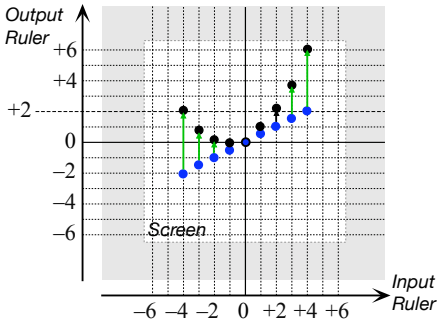
In order to get the plot for the output function $OUTPUT$ we proceed as follows:

For each input,

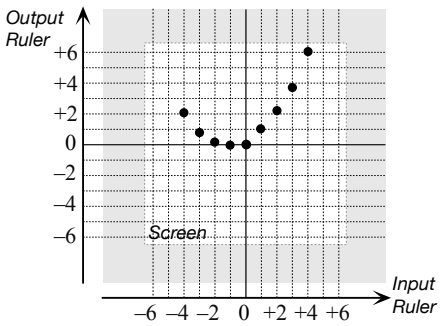
We count on the plot of $ADD\text{-}ON$ the outputs it returns



We recount these outputs on the plot of $INPUT$ starting from the outputs it returns

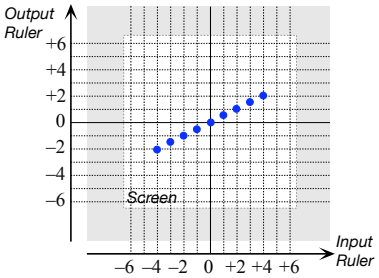


The resulting points are the plot point of $OUTPUT$

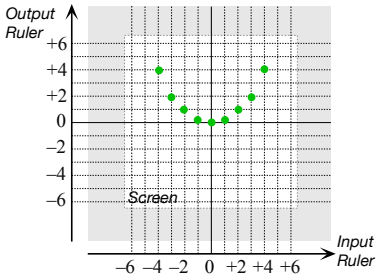


Altogether we have:

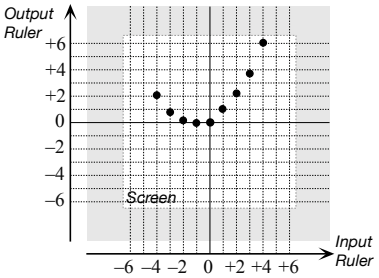
$INPUT$:



$ADD\text{-}ON$:



$OUTPUT$:



Index