Chapter 1

Relations And Functions

1.1 Relations

To see whether something is changing—or not changing, we must look at it in relationship with something else.

**Example 1.** The amount of income tax changes in relationship with the amount of income, the amount of property tax changes in relationship with the amount of property, the amount of sales tax changes in relationship with the amount of purchases.

More precisely, in order to observe something that is changing—or not changing, we must pair each observed state with some reference state.

**Example 2.** We might say that someone’s income tax was $2,753.

But, by itself, that would not be saying much since, for instance, $2,753 was a lot less money in FY 2013 (FY is for Fiscal Year) than it was in, say, FY 1913—the year income tax was first established.

So, in order for $2,753, the observed state, to make sense, we must give it along with the reference state, namely the Fiscal Year.

We will call relation whatever process, device, procedure, agency, converter, exchanger, translator, etc in terms of which the pairing is done and:

- **input numbers** will be the numbers that represent the reference states
- **output numbers** will be the numbers that represent the observed states.
CHAPTER 1. RELATIONS AND FUNCTIONS

**NOTE.** The words *input* and *output* are taken from computer science because they are more suggestive than the more traditional words, *point* for input and *value* for output. An **input-output pair** is then an *input number* together with an *output number* that the input number is paired with by the relation. Input-output pairs must be enclosed within a pair of parentheses.

**EXAMPLE 3.** In Example 2, (FY2003, $2,753) is an **input-output pair**.

### 1.2 Quantitative Rulers

For the purpose of picturing the *input numbers* and the *output numbers* involved in a given relation we will use **quantitative rulers** which are just what goes in the real world by the name of “ruler”.

**NOTE.** In high school, **quantitative rulers** usually go by the name of **number line**, but we prefer to stick with the real world.

1. The **tickmarks** on a quantitative ruler must be **labeled**, **in order**, **evenly spaced** and must include a tickmark for 0.

**EXAMPLE 4.** The following:

1. Are both quantitative rulers but the following is not a quantitative ruler even though the tickmarks are **labeled**, **in order** and evenly spaced because it does not include a tickmark for 0.

2. In other words, what specifies a **quantitative ruler** is:
   - The **extent** of the quantitative ruler, that is both the **smallest label** and the **largest label** which we will write between **square brackets** [ , ]

---

1. Actually, never ones to shy away from the impenetrable, or perhaps just to show off, Educologists generally prefer the XIX\textsuperscript{th} century terms, “independent variable” and “dependent variable”.
2. It would be interesting to trace the origin of this remarkably un-enlightening term. But then, it is probably due to Educologists’ well known craving for the esoteric.
1.2. QUANTITATIVE RULERS

- The resolution of the quantitative ruler, that is the distance between the labels of any two consecutive tickmarks.

**Example 5.** Given the following quantitative ruler

- the extent of the given quantitative ruler is \([-40, +80]\],
- the resolution of the given quantitative ruler is 10.

3. From the point of view of the extent of a given quantitative ruler, there are two kinds of numbers:
- The numbers that fall within the extent of the quantitative ruler. We will call these numbers **finite numbers** except for 0 which will just be ... 0. (We will see why in the next chapter.)
- The numbers that fall beyond the extent of the quantitative ruler. We will call these numbers **infinitely large numbers**.

**Example 6.** Relative to the following quantitative ruler:

the number +308 195 is a **finite number** but the number −208 195 is an **infinitely large number**.

4. From the point of view of the resolution of a given quantitative ruler, there are two kinds of numbers:
- The **finite** numbers that fall on a tickmark. We will call these finite numbers **tickmarked finite numbers**.
- The **finite** numbers that fall between tickmarks. We will call these finite numbers **un-tickmarked finite numbers**.

**Example 7.** Given the following quantitative ruler

the **finite numbers** −1.4 and −0.8 are **tickmarked numbers** but the **finite number** −1.1 is **not** a tickmarked number.

5. We can then plot any particular tickmarked finite number by placing a **solid dot** on the corresponding tickmark. We will then call it a **plotted number**. We cannot plot un-tickmarked finite numbers.

**Example 8.** Given the following quantitative ruler

the finite numbers −500 and +1,300 are **marked finite numbers** and therefore can be plotted.
1.3 Neighborhoods

In order to deal with un-tickmarked numbers, we will look at them in relation to the closest tickmarked number. More precisely,

i. We will look at each *tickmark* on a *quantitative ruler* as the **center** of a *neighborhood* extending between the two *halfway-marks* surrounding the center. We will draw the neighborhood within parentheses.

**Example 9.** Given the quantitative ruler

![Ruler with halfway marks](image)

the half-way marks are

and, for instance, the neighborhood with center $-2$ extends between $-1.5$ and $-2.5$:

![Neighborhood of -2](image)

ii. Any given *finite number* that is *not* a tickmarked number will however fall in the *neighborhood* of the closest tickmarked number and will then say that the *finite number* is **near** the tickmarked number.

**Example 10.** Given the following quantitative ruler

![Ruler with halfway marks](image)

the finite number $-1.57$ is un-tickmarked. The closest tickmarked number is $-1.6$ and the *neighborhood* of $-1.6$ is

![Neighborhood of -1.6](image)

So, we will say that $-1.57$ is in the *neighborhood* of $-1.6$: 

but the finite numbers $-250$ and $+800$ are not marked finite numbers and therefore cannot be plotted.
1.4 Quantitative Screens

Given a relation, we will often want to picture the input-output pairs involved in that relation.

1. The simplest way, of course, is to use:
   - a quantitative input ruler to plot the input numbers
   - a quantitative output ruler to plot the output numbers
   - and links to show input-output pairs.

**Example 11.** We could picture the input-output pair (FY1961, $2,770), as follows:

Obviously, though, this approach is not going to work very well when the relation involves many input-output pairs.

2. For a better way to picture the input-output pairs of a relation, we will use a quantitative screen that consists of:
   - A screen, that is a rectangular area in which we will picture the input-output pairs,
   - Some space around the screen which we will call offscreen,
   - A quantitative input ruler placed under the screen with the extent of the input ruler corresponding to the width of the screen,
   - A quantitative output ruler placed left of the screen with the extent of the output ruler corresponding to the height of the screen.

**Example 12.**
3. The procedure for linking an input with an output then is:

1. To link an input with a corresponding output
   i. We plot the input number on the input ruler,
   ii. We draw the input level line, that is the vertical line through the tickmark,
   iii. We plot the output number, on the output ruler,
   iv. We draw the output level line, that is the horizontal line through the tickmark,

**EXAMPLE 13.** In order to draw the link from the input $-2$ to the output $+5$,

i. We plot the input number $-2$ on the input ruler,
ii. We draw the input level line through the tickmark,
iii. We plot the output number $+5$ on the output ruler,
iv. We draw the output level line through the tickmark,
v. The link goes from the input $-2$ to the intersection of the two level lines and then from the intersection of the two level lines to the output $+5$.

4. The plot point for an input-output pair is the “elbow” of the link, that is the intersection of the input level line with the output level line. We mark the plot point with a solid dot. In fact, with the plot point, we do not need the link because, from the plot point we can always recover the input-output pair that it is the plot-point of:
2. To recover the input-output pair from the plot point

i. We draw the input level line, that is the vertical line, through the plot point,

ii. We read the input number where the input level line intersects the input ruler,

iii. We draw the output level line, that is the horizontal line, through the plot point,

iv. We read the output number where the output level line intersects the output ruler,

**Example 14.** Given, the following plot-point

we recover the input-output pair as follows:

i. We draw the input level line through the plot point,

ii. We read the input number where the input level line intersects the input ruler,

iii. We draw the output level line through the given plot point,

iv. We read the output number where the output level line intersects the output ruler,

In other words, a plot point does not cause any loss of information compared to the input-output pair that it pictures.
1.5 Functions

Perhaps surprisingly, relations can get to be extremely complicated and so, from now on, we will only investigate functions, that is relations that meet the requirement that:

\[ \text{No input shall be paired with more than one output.} \]

In other words, given any input, a function may either return one output or no output at all but never more than one output.

However, there is nothing to prevent a function from pairing many inputs with a same, single output. In other words, a function may return the same output for different inputs.

**Example 15.** A parking meter is a function because, whatever the amount of money we input, we can be sure that anyone who inputs the same amount of money will get the same amount of parking times.

However, since there usually is a maximum parking time, any amounts of money we input above the maximum will return the same amount of parking time namely the maximum parking time.

**Example 16.** A slot machine is not a function because, given an input, say 1 Quarter, a slot machine can output just about any number of Quarters (depending on our luck).

**Example 17.** We would like the relation between income and income tax to be a function because, presumably, we don’t want two persons with the same income to pay different amounts of income tax! But then that’s why we pay taxes on “net income” rather than “gross income”.

---

3Educologists will rightfully object that functions should not be allowed to return no output and therefore that we should introduce the notion of domain. But while, of course, pontificating about domains can be very gratifying to the instructor, it also complicates the students’ mathematical life quite unnecessarily since, at this point, the difficulty this causes is quite unlikely to occur with beginners.
1.6 Functions Specified By A Global I-O Rule

Functions can be specified in several ways:

1. In some sciences, such as Psychology, Sociology, Business, Accounting, etc functions are usually specified by input-output table(s). We will refer to them as tabular functions.

**Example 18.** A business may be described by its profits/losses over the years, that is by a tabular function specified by the following input-output table:

<table>
<thead>
<tr>
<th>Fiscal Year</th>
<th>Profit/Loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>2001</td>
<td>+5,924</td>
</tr>
<tr>
<td>2002</td>
<td>-2,351</td>
</tr>
<tr>
<td>2003</td>
<td>+6,753</td>
</tr>
<tr>
<td>2004</td>
<td>+5,636</td>
</tr>
<tr>
<td>2005</td>
<td>-3,753</td>
</tr>
<tr>
<td>2006</td>
<td>+8,482</td>
</tr>
</tbody>
</table>

2. In this text, we will investigate only functions specified by a global input-output rule constructed as follows:

i. We must have a name for the function. (The letter $f$ is often used.)

ii. We use a letter, usually $x$, which we will call unspecified input, to stand for the input until a specific input is given. In other words, we will be able at any time to replace the unspecified input $x$ by any specific input we want.

iii. Then, $f(x)$, to be read as “$f$ of $x$”, yet another use of parentheses, will stand for the output returned by the function $f$ for the unspecified input $x$.

iv. Finally, we must give an output specifying code to specify $f(x)$ in terms of $x$.

Altogether then, a global input-output rule will look like this

\[ x \rightarrow f(x) = \text{Output-specifying code} \]

with the understanding that whatever specific input we replaced the unspecified input $x$ with will also be used to replace $x$ in the output-specifying code for $f(x)$.
CHAPTER 1. RELATIONS AND FUNCTIONS

**Example 19.** Let *JILL* be the function specified by the global input-output rule

\[
x \xrightarrow{\text{JILL}} JILL(x) = \frac{2\sin x + 5}{-5x^3 + 2\exp x}
\]

The code \(\frac{2\sin x + 5}{-5x^3 + 2\exp x}\) (which at this point we do not know how to read) is what specifies the output *JILL*(*x*) in terms of the input *x*.

3. Let *f* be the function specified by the global input-output rule

\[
x \xrightarrow{\text{f}} f(x) = \text{Output-specifying code}
\]

and let \(x_0\) be a specified input. Then, the procedure for getting \(f(x_0)\), namely the output returned by the function *f* for the specified input \(x_0\), is:

3. **Output returned for a specified input \(x_0\) by a function specified by an I-O rule.**

i. We **declare** that \(x\) is to be replaced by the specified input \(x_0\).

\[
x \bigg|_{x=x_0} \xrightarrow{\text{\(f\)}} f(x) \bigg|_{x=x_0} = \text{Output-specifying code} \bigg|_{x=x_0}
\]

Instructions for what to do with \(x_0\) to get \(f(x_0)\)

ii. We **decode** the output specifying code, that is we write out the computations to be performed according to the output-specifying code.

iii. We **perform** the computations specified by the code and thus get the output \(y_0\).

iv. We **format** the input-output link according to our purpose:

- **Computational purpose**: \(f(x_0) = y_0\)
- **Functional purpose**: \(x_0 \xrightarrow{\text{\(f\)}} y_0\)
- **Graphic purpose**: \((x_0, y_0)\) is a plot-point for *f*

**Example 20.** Let *JACK* be the function specified by the global input-output rule

\[
x \xrightarrow{\text{JACK}} JACK(x) = -4 \otimes x \oplus +7
\]

Let the specified input be \(-3\). Then, to get the output returned for the specifies input \(-3\) by the function *JACK*, we proceed as follows:
1.7. FUNCTIONS SPECIFIED BY A CURVE

Any curve we draw on a quantitative screen specifies a relation because each point on the curve can be looked upon as being a plot point from which we can recover the input-output pair as we did above.

1. If it happens that no input-level line intersects the curve in more than one point, then the relation meets the requirement that no input shall be paired with more than one output.
since none of input-level lines intersects the curve in more than one point, the curve specifies a function whose quantitative finite graph is:

2. When a function is specified by a curve, then, given an input, we get the output for that input as follows:

4. Output returned for a specified input \(x_0\) by a function specified by a curve.

i. We plot the input number on the input ruler,

ii. We draw the input level number through the input number,

iii. We mark the intersection of the input level line with the curve with a plot point,

iv. We draw the output level line through the plot point,

v. Then, the output number is where the output level line intersects the output ruler.

**Example 22.** Let \(NANCY\) be the function specified by the quantitative global
Let the specified input be $-2$. Then, to get the output returned by NANCY.

i. We plot the input number $-2$ on the input ruler,

ii. We draw the input level line through the input number

iii. We mark the intersection of the input level line with the curve with a plot point,

iv. We draw the output level line through the plot point,

v. Then, the output number is where the output level line intersects the output ruler.

3. A function being specified by a global input-output rule, we will usually want, for various reasons, to find its quantitative finite graph.

One reason is that it is usually impossible to see just from the global input-output rule which inputs have some required “feature”. But with the finite global graph, it will often be obvious which input(s), if any, have such a required “feature”.

**Example 23.**

Given the function MILT specified by the quantitative global graph
find the inputs whose output is less than 2. From the quantitative finite graph,

see that the answer is “All inputs between $-4$ and $+3$”

However, getting the quantitative global graph of a function specified by a global input-output rule is usually not a simple matter.

1.8 “Joining Plot Points Smoothly”

In high school, to get the quantitative global graph of a function specified by a global input-output rule, we were taught to proceed as follows:

i. Create an input-output table by picking a few inputs and using the global input-output rule to compute the corresponding outputs.

ii. Plot the input-output table.

iii. “Join smoothly” the plot-points.

1. The trouble with that supposed “procedure” is that, while there is no problem with the first two steps, we have absolutely no reason whatsoever to believe that the curve we get in the third step will be anywhere like the actual quantitative finite graph of the function.

Indeed, given a number of plot-points, with the single exception of affine functions which we will discuss in Chapter 10, there is never a unique way
1.8. “JOINING PLOT POINTS SMOOTHLY”

to “join the plot points smoothly”

**Example 24.** Given a function $RAT$, suppose we have obtained the following table:

<table>
<thead>
<tr>
<th>Inputs</th>
<th>$-4$</th>
<th>$-3$</th>
<th>$-2$</th>
<th>$+1$</th>
<th>$+2$</th>
<th>$+4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outputs</td>
<td>$-1$</td>
<td>$+3$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$-2$</td>
<td>$+3$</td>
</tr>
</tbody>
</table>

and therefore the following plot:

All six of the following graphs come from “joining smoothly” the above plot points:

---

4This is perhaps where the dishonesty of Educologists is most patent as one would expect them at least to acknowledge this rather unfortunate fact. But they never do.
2. The advice we were usually given at this point was to “get more plot points” but this raises in turn two issues:

- One issue is the question of how many plot-points would be needed to guarantee that there is only one way to joint the plot points to get the graph. As you will see as you go through this text, there generally isn’t any such number.

**Example 25.** Given the function $CAT$ specified by the global input-output rule

$$x \xrightarrow{CAT} CAT(x) = \frac{x^3 - 1}{x - 2}$$

here are six computer generated plots with so many plot points that they all look like quantitative finite graphs:
The question is: Which should we decide is the quantitative finite graph?

- The other issue is that having very many plot points can turn out to make it impossible to “join them smoothly”.

**Example 26.** The function $SINE$, which is of major importance in Physics, Engineering, etc, is specified by a “functional equation” rather than by a global input-output rule. As such, it doesn’t belong to this text, but what matters here is Strang’s Famous Computer Plot of $SINE$: 

What is the quantitative finite graph?

What made Strang’s computer plot famous is that the quantitative finite graph of $SINE$ is in fact a lot simpler than the above computer plot would suggest.
1.9 Qualitative Global Graph

In any case, the immense variety of possible global input-output rules would seem to make finding a “universal procedure” for getting a quantitative finite graph for every function specified by a global input-output rule most unlikely.

Therefore, our goal in this text will be a lot more modest: instead of the quantitative viewpoint we took so far, we will from now on take a qualitative viewpoint and we will go only for qualitative global graphs, that is global graphs that might give us only a partial idea of the way the quantitative global graph really looks.

But, even as such, qualitative global graphs are necessary when “joining smoothly” plot points and, much more importantly and as will be the emphasis in this text, qualitative global graphs will turn out to be a crucial tool when investigating the existence of inputs with required “features”.

And so, after we have developed some “technology” in the next two chapters, our primary goal in the rest of this text will be to investigate, in the case of gradually more complicated functions, the

**FUNDAMENTAL PROBLEM:** Given a function specified by a global input-output rule, get a qualitative global graph.
Index

[ ], 2
brackets, square, 2
center, 4
declare, 10
decode, 10
distance, 3
domain, 8
dot, solid, 3
evenly spaced, 2
extent, 2
finite number, 3
format, 10
function, 8
halfway-mark, 4
in order, 2
input level line, 6
input number, 1
input-output pair, 2
input-output rule, global, 9
labeled, 2
large infinitely number, 3
link, 5
loss of information, 7
near, 4
neighborhood, 4
number line, 2
observe, 1
observed state, 1
offscreen, 5
output level line, 6
output number, 1
output specifying code, 9
pair, 1
perform, 10
plot, 3
plot point, 6
plotted number, 3
qualitative global graphs, 18
quantitative finite graph, 11
quantitative input ruler, 5
quantitative output ruler, 5
quantitative screen, 5
reference state, 1
relation, 1
resolution, 3
return, 8
ruler, quantitative, 2
screen, 5
solid dot, 6
specific input, 9
specified, 9
specified input, 10
tabular function, 9
tickmarked finite number, 3
tickmarks, 2

un-tickmarked finite number, 3
unspecified input, 9