Chapter 1

Qualitative Numbers

We begin by revisiting a number of concepts which the reader already encountered in some form or the other but which s/he should nevertheless carefully study here, “pencil in hand”, because they are discussed in precisely the way they will be needed for the qualitative analysis of “functions” which is the purpose of this text. (The concept of function itself will be introduced in Chapter 2.)

1.1 Sign and Size

A signed number carries two very different pieces of information:

- a sign, namely + or −, which is the qualitative part of a “feature” in that it indicates “which way” the feature is going.
  
  **NOTE.** In this text, the sign + will never “go without saying”. In other words, we will always distinguish positive numbers such as +5 from plain numbers such as 5.

- a size, namely a plain number, also known as unsigned number, which is the quantitative part of the feature in that it indicates “how much” of the feature there is.
  
  **NOTE.** For this concept, textbooks mostly use the words “absolute value” or, sometimes, “numerical value”, “modulus”, or “norm”. None of these will be used...
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Say the signed number $+17.43$ represents a real-world money transaction or a real-world slope. Then,

- **Sign** of $+17.43 = +$ indicates that the money transaction was a gain or that the slope is upward.
- **Size** of $+17.43 = 17.43$ (a plain number) indicates how much money was transacted or how steep the slope is.

1.2 Quantitative Rulers

To picture the specific signed numbers involved in a given situation, we will use quantitative rulers which are indeed essentially what goes in the real world by the name of “ruler”.

1. More precisely, the tickmarks on a quantitative ruler must be labeled, in order, evenly spaced and must include an origin, that is a tickmark labeled 0.

**NOTE.** In high school, quantitative rulers go by the name of number line.\(^1\)

**EXAMPLE 2.** The following:

$$
\begin{array}{cccccccccc}
\text{Ruler} & \text{Ruler} \\
-8000 & -7000 & -6000 & -5000 & -4000 & -3000 & -2000 & -1000 & 0 & +1000 \\
\end{array}
$$

are both quantitative rulers but the following is not a quantitative ruler because, even though the tickmarks are labeled, in order and evenly spaced, there is no origin.

$$
\begin{array}{cccccccccc}
\text{Ruler} & \\
-100 & -50 & 50 & 100 & 150 & 200 & 250 & 300 & 350 & 400 \\
\end{array}
$$

and the following is not a quantitative ruler even though the tickmarks are labeled and in order because the tickmarks are not evenly spaced:

$$
\begin{array}{cccccccccc}
\text{Ruler} & \\
-8000 & -60 & -4000 & -2000 & -1000 & 0 & +1000 & +153.72 \\
\end{array}
$$

2. From the graphic viewpoint:

- The **sign** of a signed number codes which side of 0 the signed number is on a quantitative rule. Inasmuch as, as will be the case here, the labels to the left of 0 are negative and the labels to the right of 0 are positive,
  * The sign $+$ says that the signed number is to the **right** of 0,
  * The sign $-$ says that the signed number is to the **left** of 0.

\(^1\)It would be interesting to trace the origin of this remarkably un-enlightening term. But then, it is probably just due to Educologists’ well known craving for the esoteric.
**Example 3.** On a quantitative ruler,

Since $\text{Sign of } -5 = -$, the signed number $-5$ is left of 0.

Since $\text{Sign of } +3 = +$, the signed number $+3$ is right of 0.

- The size of a signed number codes how far from 0 the signed number is on a quantitative ruler.

**Example 4.** The signed numbers $-5$ and $+5$ have the same size, namely 5, so they are equally far from 0:

3. Quantitative rulers are specified by:

- The extent of the quantitative ruler which consists of the smallest label together with the largest label. Extents will be coded between square brackets $[\ , \ ]$.

- The resolution of the quantitative ruler which is the plain number that is the distance between any two consecutive labels.

**Example 5.** Given the following quantitative ruler

- the extent is $[-40, +80]$,
- the resolution is 10.

4. From the point of view of the extent, there are two kinds of numbers:

- The numbers that fall within the extent of the quantitative ruler. We will call these numbers finite numbers except for 0 which is not a finite number but just ... zero. (More about zero in section 1.6.)

- The numbers that fall beyond the extent of the quantitative ruler. We will call these numbers infinitely large-in-size number. (More about infinitely large-in-size numbers in section 1.7.)

**Example 6.** Relative to the following quantitative ruler:

the number $+308\,195$ is a finite number and the number $-208\,195$ is an infinitely large-in-size number.
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5. From the point of view of the resolution, there are two kinds of finite numbers:
   - The finite numbers that fall on a tickmark. We will call these finite numbers **tickmarked finite numbers**.
   - The finite numbers that fall between tickmarks. We will call these finite numbers **non-tickmarked finite numbers**.

**Example 7.** Given the following quantitative ruler

```
-4 -3 -2 -1 0 +1 +2 +3
```

-1.4 and -0.8 are **tickmarked** finite numbers but -1.1 is a **non-tickmarked** finite number.

6. We can **plot** any tickmarked finite number by placing a **solid dot** on the corresponding tickmark. We will then call that tickmarked finite number a **plotted number**. We cannot plot non-tickmarked finite numbers.

**Example 8.** Given the following quantitative ruler

```
-600 -400 -200 0 +200 +400 +600 +800 +1000 +1200 +1400 +1600 +1800
```

+2 000 is an **infinitely large-in-size** number and so cannot be plotted. -600 and +1 400 are **tickmarked** finite numbers and therefore can be plotted:

```
-600 -400 -200 0 +200 +400 +600 +800 +1000 +1200 +1400 +1600 +1800
```

but -250 and +700 are **non-tickmarked** finite numbers and so cannot be plotted.

1.3 Neighborhoods and Nearness

In order to deal with non-tickmarked finite numbers, we will look at them in relation to the closest tickmarked number. More precisely,

1. We will look at each tickmark on a quantitative ruler as the **center** of a **neighborhood** extending between the two **halfway-marks** surrounding the center. We will draw the neighborhood within parentheses.

**Example 9.** Given the quantitative ruler

```
-4 -3 -2 -1 0 +1 +2 +3
```

the halfway marks are

```
Halfway marks
```

and, for instance, the neighborhood with center -2 extends between -1.5 and -2.5:
2. Any non-tickmarked finite number will then be in the neighborhood of the closest tickmarked finite number and we will say that the non-tickmarked finite number is near the tickmarked finite number.

**Example 10.** Given the following quantitative ruler

![Quantitative Ruler]

the finite number $-1.57$ is non-tickmarked. The closest tickmarked finite number is $-1.6$ and since the neighborhood of $-1.6$ is

![Neighborhood of -1.6]

the number $-1.57$ is in the neighborhood of $-1.6$:

![Neighborhood of -1.6 -1.57]

or just that $-1.57$ is near $-1.6$

3. Numbers that are near 0, that is numbers that are in the neighborhood of zero, will be called infinitely small-in-size numbers. (More about infinitely small-in-size numbers in section 1.6.)

**Example 11.** Both $-0.056$ and $+0.19$ are infinitely small-in-size numbers.

1.4 Qualitative Rulers

1. For the purpose of qualitative analysis, more precisely for the purpose of visualizing features of functions, we will use qualitative rulers that is rulers with only one tickmark for 0 and an arrowhead to indicate which way is up and a tickmark for whatever finite number we happen to be investigating.

**Example 12.** If we are interested in investigating the number $+5.2$, we can use the following picture involving a qualitative ruler:

![Qualitative Ruler]
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1. The concept of neighborhood and the way we picture a neighborhood will remain unchanged on qualitative rulers.

Example 13. Let us say that we are interested in the numbers that are near 5.2.

We will represent the neighborhood of 5.2 as follows:

3. While the concept of resolution disappeared the concept of extent remains and we will continue to distinguish:
   • The numbers that fall within the extent of the qualitative ruler which we will continue to call finite number—except for the number 0 which we will continue to call zero,
   • The numbers that fall beyond the extent of the qualitative ruler which we will continue to call infinitely large-in-size numbers.
   • The numbers which are near 0 which we will continue to call infinitely small-in-size numbers

So, graphically, we will continue to have:

1.5 Comparisons of Signed Numbers

Before we can return to infinitely large-in-size numbers and infinitely small-in-size numbers, we need to discuss the two ways signed numbers can be compared.

1. When we compare two plain numbers, we can characterize the larger number and the smaller number either one of two ways, that is we can either say that
   • The larger plain number is the one that is farther from 0 and the smaller plain number is the one that is closer to 0.
   or, with the usual agreement that the way up on the ruler (as indicated by the arrowhead) is from left to right, we can say that
1.5. COMPARISONS OF SIGNED NUMBERS

- The larger plain number is the one that is rightmost and the smaller plain number is the one that is leftmost.

Indeed, for plain numbers, the two ways always give the same answer.

**Example 14.**

![Plain Ruler Diagram]

2. When we compare signed numbers however, the two ways do not give the same answer when the two numbers have opposite signs.

**Example 15.** Farther from 0 need not be rightmost:

![Signed Ruler Diagram]

We therefore have two different ways to compare given two signed numbers:

- The signed number that is larger is the one that is rightmost (with the usual agreement that the way up is from left to right) and the signed number that is smaller is the one that is leftmost.

**Example 16.** Given the signed numbers $-7$ and $-1$:

![Signed Ruler Diagram]

since $-1$ is rightmost, the signed number $-1$ is the larger one and since $-7$ is leftmost, the signed number $-7$ is the smaller one.

- The signed number that is larger-in-size is the signed number that is farther from 0 and the signed number that is smaller-in-size is the signed number that is closer to 0.

**Example 17.** Given the signed numbers $-4$ and $+3$:

![Signed Ruler Diagram]

since $+3$ is closer from 0, the signed number $+3$ is the smaller-in-size and since
1.6 Zero and Infinitely-Small Numbers

That 0 is not a number like any other can already be seen from the fact that it is a late invention (9th century AD) compared to the invention of the other numbers. [http://en.wikipedia.org/wiki/0_%28number%29#History](http://en.wikipedia.org/wiki/0_%28number%29#History).

1. The real issue is that 0 as a number in connection with *collections* is one thing while 0 as a number in connection with *quantities* is quite a different thing:
   - In connection with real-world *collections*, 0 is used to represent *empty* collections.
     
     **Example 18.** Just as the number phrase *3 Apples* represents on paper the real-world collection [🍎🍎🍎], in the same manner the number phrase *0 Apples* represents on paper the real-world *empty* collection [🍎].
   - In connection with real-world *quantities*, 0 does not correspond to anything.
     
     **Example 19.** Nothing is at a temperature of exactly 0°Kelvin, nothing is exactly 0mm thick nor is there any perfect vacuum, etc.

And, of course, inasmuch as we are Calculus-oriented, we are concerned with *quantities*!

2. What is actually used in place of 0 in science, technology and engineering are *infinitely small-in-size* numbers, that is decimal numbers starting with lots of 0’s.

**Example 20.** The use of laser cooling has produced temperatures less than a billionth of a degree kelvin. [http://en.wikipedia.org/wiki/Absolute_zero](http://en.wikipedia.org/wiki/Absolute_zero)

3. Graphically, we look at infinitely small-in-size numbers as numbers in the neighborhood of 0, that is as numbers that are near 0, but, while the more leading 0’s there are the nearer the infinitely small number is to 0, there is no realistic way to picture exactly an infinitely small-in-size number.

**Example 21.** There is no realistic way to picture an infinitely small number such as +0.000 000 002. All we can really do is to indicate that it is in the neighborhood of 0.

4. We will often have to distinguish the two sides of the neighborhood of 0 and:
1.7. INFINITY AND INFINITELY-LARGE NUMBERS

- We will use $0^+$ to designate infinitely small-in-size numbers that are on the positive side of 0,
- We will use $0^-$ to designate infinitely small-in-size numbers that are on the negative side of 0,

**Example 22.**

![Diagram of a ruler with neighborhoods of 0, 0+, and 0- labeled]

1.7 Infinity and Infinitely-Large Numbers

When, starting from the origin, we go straight ahead we have the feeling that we can keep on going as far away as we want and that the longer we go, the farther away from the origin we get. But this is not really true. In fact, even though Magellan died in 1521 while trying to go as far away as he could from Seville, one of his ships eventually reached home, bearing witness that there was no going around the fact that there is no such thing as a straight line and that the earth is round. We will refer to a circle around the earth as a **Magellan circle**.

1. Looking at things that way, there is a point on the Magellan circle furthest away from the origin, namely the point “down under”. We will call this point **infinity** with $\infty$ the symbol for infinity.
2. When we look at a quantitative ruler as part of a Magellan circle, we see that the two stretches that are beyond the extent make up in fact a single stretch of the Magellan circle whose center is $\infty$. We will say that this stretch is a **neighborhood of** $\infty$. And of course we will say that the infinitely large numbers, that is the numbers that are beyond the extent, are in a neighborhood of $\infty$ and that they are **near** $\infty$.

**NOTE.** The reason the parentheses at the end of the extent face $\infty$ is because the parentheses **encompass a neighborhood** of $\infty$.

3. Very often, just the way we will need to distinguish the two sides of 0, we will need to distinguish the two side of $\infty$. But in the case of $\infty$, there are two ways to do so:

i. One way is just to use the side of $\infty$ that the numbers are on and to distinguish:

- The numbers that are **left of** $\infty$
- The numbers that are **right of** $\infty$

**EXAMPLE 23.** The sides of $\infty$ are clear on a Magellan circle:

$$
\begin{array}{c}
\text{Infinity}\\
\text{LEFT}\\
\text{RIGHT}\\
-\infty +170 \text{ is left of } \infty \\
-110 \text{ is right of } \infty \\
+\infty
\end{array}
$$

so that

These **infinitely large-in-size numbers** are right of $\infty$.

These **infinitely large-in-size numbers** are left of $\infty$.

It is important, though, to realize that:

- The numbers that are to the **left of** $\infty$ are to the **right of** 0
- The numbers that are to the **right of** $\infty$ are to the **left of** 0

**EXAMPLE 24.**
ii. The other way is to use the sign of the numbers and to distinguish between

- The numbers to the right of the extent are on the positive side of $\infty$ and we say that the numbers are near $+\infty$
- The numbers to the left of the extent are on the negative side of $\infty$ and we say that the numbers are near $-\infty$

Example 25.

These infinitely large-in-size numbers are near $-\infty$
These infinitely large-in-size numbers are near $+\infty$

and therefore, from the qualitative viewpoint,

These infinitely large-in-size numbers are near $-\infty$
These infinitely large-in-size numbers are near $+\infty$

1.8 Computational Definitions

1. While the concepts of smaller-in-size is quite clear, the perhaps apparently simpler concept of infinitely small-in-size is in fact not all that simple to pin down.

   a. Up to a point, infinitely small-in-size is a “relative concept”. Nevertheless, infinitely small-in-size has a meaning which is the same for everybody and it is only the cutoff size that changes depending on the situation and from people to people.
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computational definition

**Example 26.** Nobody likes to work for a small amount of money: people like the Koch brothers would not dream to work a whole day for ten-thousand dollars any more than “the rest of us” would like to work a whole day for ten dollars.

b. However, we will need a computational definition, that is a definition that we can compute with, and we will say that:

A number is **infinitely small-in-size** if copies of that number will multiply to a number that is smaller-in-size than the number itself.

**Example 27.** $-0.2$ is *small-in-size* because, say, two copies of $-0.2$ multiply to $+0.04$ which is smaller-in-size than $-0.2$.

c. In practice, though, whenever we will have to pick a number that is small-in-size to test something, we will leave ourselves a “safety margin” and we will just pick numbers like $\pm0.1$, $\pm0.01$, $\pm0.001$, etc.

2. While the concept of larger-in-size is quite clear, the perhaps apparently simpler concept of **infinitely large-in-size** is in fact not all that simple to pin down.

a. Up to a point, *infinitely large-in-size* is a “relative concept”. Nevertheless, *infinitely large-in-size* has a meaning which is the same for everybody and it is only the cutoff size that changes depending on the situation and from people to people.

**Example 28.** A million dollars, whether gained or lost, is probably not a large amount of money for people like the Koch brothers but it surely is for “the rest of us”. On the other hand, even for the Koch brothers, a billion dollars is a large amount to gain or to lose.

b. However, we will need a computational definition, that is a definition that we can compute with, and we will say that:

A number is **infinitely large-in-size** if copies of that number will multiply to a number that is larger-in-size than the number itself.

**Example 29.** $-1.1$ is infinitely large-in-size because, say, three copies of $-1.1$ multiply to $-1.331$ which is larger-in-size than $-1.1$.

c. In practice, though, whenever we will need to pick a number that is infinitely large-in-size to test something in a computation, we will leave ourselves a “safety margin” and so computationally, we will pick numbers like $\pm10$, $\pm100$, $\pm1000$, etc.

3. Finite numbers are neither infinitely small-in-size nor infinitely large-in-size.
EXAMPLE 30. +1 is neither infinitely small-in-size nor infinitely large-in-size because copies of +1 will multiply to +1 whose size is 1, the same as the size of +1. Similarly, −1 is neither infinitely small-in-size nor infinitely large-in-size because copies of −1 will multiply to +1 or −1 depending on whether the number of copies is even or odd but, in both cases, the size of the result will be 1, the same as the size of −1.

In practice, the finite numbers will be those numbers whose size is between 0.1 and 10 except of course those for those whose size is near 0.

4. Finally:

**NOTE.** In this text, we will be using the concepts of infinitely small-in-size and infinitely large-in-size so often that, because we are lazy, we will let the words “infinitely” and “in-size” go without saying and, from now on, we will use:

- the shorter “small” when in fact we will mean the longer “small-in-size”
- the shorter “large” when in fact we will mean the longer “large-in-size”.

5. So, altogether and to recap, we will be dealing with three kinds of numbers:

- **finite numbers** are between **large numbers** and **small numbers**

1.9 Multiplication and Division of Signs

For reference, we recall here the Rule for Multiplication and Division of Signs:

<table>
<thead>
<tr>
<th></th>
<th>+</th>
<th>−</th>
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<tbody>
<tr>
<td>+</td>
<td>+</td>
<td>−</td>
</tr>
<tr>
<td>−</td>
<td>−</td>
<td>+</td>
</tr>
</tbody>
</table>

1.10 Multiplication and Division of Sizes

By computing with large, small and finite, we mean something similar to what we mean by computing with numbers but only up to a certain point.

1. We have the following
THEOREM 1 (Size Multiplication Theorem).

\[
\begin{align*}
\text{large} \cdot \text{large} &= \text{large} & \text{large} \cdot \text{finite} &= \text{large} & \text{large} \cdot \text{small} &= \text{?} \\
\text{finite} \cdot \text{large} &= \text{large} & \text{finite} \cdot \text{finite} &= \text{finite} & \text{finite} \cdot \text{small} &= \text{small} \\
\text{small} \cdot \text{large} &= \text{?} & \text{small} \cdot \text{finite} &= \text{small} & \text{small} \cdot \text{small} &= \text{small}
\end{align*}
\]

The case for the Size Multiplication Theorem goes as follows:

i. Since \text{large}, \text{small} and \text{finite} were defined in terms of multiplying copies, the non-highlighted entries are pretty much as we would expect.

\textbf{EXAMPLE 31.} To find the size of \text{large} \cdot \text{large}, we multiply two copies of \text{large} \cdot \text{large}:

\[
(\text{large} \cdot \text{large}) \cdot (\text{large} \cdot \text{large}) = \text{large} \cdot \text{large} \cdot \text{large} \cdot \text{large}
\]

By the computational definition of \text{large}, four copies of \text{large} multiply to something larger-in-size than \text{large}, and something larger-in-size than \text{large} is \text{large}.

ii. The two highlighted entries \text{large} \cdot \text{small} and \text{small} \cdot \text{large} are \textit{undetermined} because the result could be \text{large}, \text{small} or \text{finite} depending on how small is \text{small} compared to how large is \text{large}.

\textbf{EXAMPLE 32.} These are instances of \text{large} \cdot \text{small} that are \textit{different-in-size}:

\[
1000 \cdot \frac{1}{10} = 100, \quad 1000 \cdot \frac{1}{1000} = 1, \quad 1000 \cdot \frac{1}{100000} = \frac{1}{100}
\]

\textbf{EXAMPLE 33.} These are instances of \text{small} \cdot \text{large} that are \textit{different-in-size}:

\[
\frac{1}{100000} \cdot 10000 = 100, \quad \frac{1}{1000} \cdot 1000 = 1, \quad \frac{1}{1000} \cdot 10 = \frac{1}{100}
\]

2. We have the following

THEOREM 2 (Size Division Theorem).

\[
\begin{align*}
\text{large} & = ? & \text{large} \div \text{large} & = \text{large} & \text{large} \div \text{small} & = \text{large} \\
\text{finite} & \div \text{large} & = \text{small} & \text{finite} \div \text{finite} & = \text{finite} & \text{finite} \div \text{small} & = \text{large} \\
\text{small} & \div \text{large} & = \text{small} & \text{small} \div \text{finite} & = \text{small} & \text{small} \div \text{small} & = ?
\end{align*}
\]

The case for the Size Division Theorem goes as follows.

i. The non-highlighted entries follow from the Size Multiplication Theorem and the fact that \(\frac{1}{\text{large}} = \text{small}\) and \(\frac{1}{\text{small}} = \text{large}\).
ii. The two highlighted entries, \( \frac{\text{large}}{\text{large}} \) and \( \frac{\text{small}}{\text{small}} \), are undetermined because the result could be large, small or finite depending on how small \( \text{small} \) is compared to how large \( \text{large} \) is.

**Example 34.** These are instances of \( \frac{\text{large}}{\text{large}} \) that are different-in-size:

\[
\frac{1\,000}{10} = 100, \quad \frac{1\,000}{1\,000} = 1, \quad \frac{1\,000}{100\,000} = \frac{1}{100}
\]

**Example 35.** These are instances of \( \frac{\text{small}}{\text{small}} \) that are different-in-size:

\[
\frac{1\,000}{1\,000} = 1, \quad \frac{1\,000}{1\,000} = 1, \quad \frac{1\,000}{100\,000} = \frac{1}{100}
\]
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