addition formula for squares
computational approach
graphic approach

Chapter 23

Addition Formulas

23.1 Dimension $n = 2$: $(x_0 + h)^2$ (Squares)

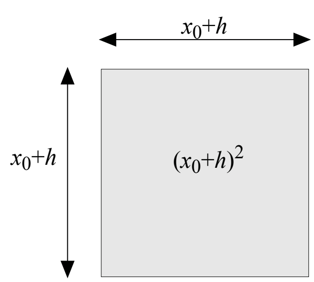
In order to get the *local input-output rule* of a given quadratic function near a given *bounded input* x_0, we will need a formula that gives us $(x_0 + h)^2$ in terms of x_0 and h and which we will call the **addition formula for squares**. We prefer to establish it here, ahead of time, rather than in the midst of the developing the local input-output rule. There are two approaches.

1. The **computational approach**. We "just" multiply two copies of $(x_0 + h)$:

$$\begin{array}{ccccc} & & x_0 & + & h \\ & & x_0 & + & h \\ \hline & & x_0h & + & h^2 \\ x_0^2 & + & x_0h & & \\ \hline x_0^2 & + & 2x_0h & + & h^2 \end{array}$$

2. The **graphic approach**.

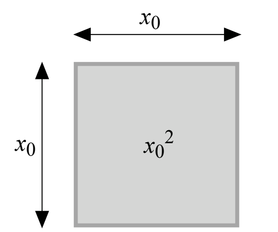
We go back to the real world and to the *definition of multiplication* in terms of the *area of a rectangle* so that $(x_0 + h)^2$ is the area of a $x_0 + h$ by $x_0 + h$ square:



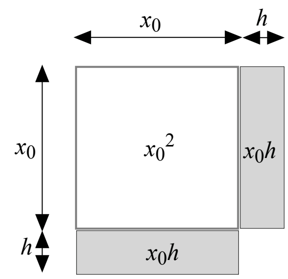
What we then do is get this area by starting with an x_0 by x_0 square, the initial square, and then augment the sides of the initial square by h and see efficient

how the area increases. For the sake of clarity, we will augment the square one step at a time:

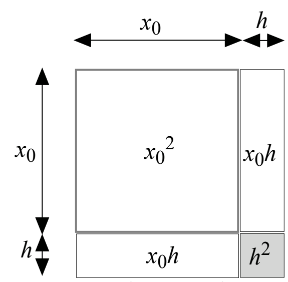
i. The sides of the initital square are equal to x_0 and the area of the initital square is therefore x_0^2:



ii. We now augment the sides of the initial square by h in each dimension which adds two x_0+h by h rectangles to the initial square:



ii. To complete the enlarged square, we must add one h by h square to what we already have::



Altogether, the area of the *augmented square*, whose side is x_0+h, is therefore:

$$x_0^2 + 2x_0h + h^2$$

3. Even though the *graphic approach* appears much more time-consuming that the *computational approach*, the graphic approach:

- Gives us the succeeding terms in *diminishing order of sizes*:
  i. Since, x_0, the side of the initial square, is *bounded*, the area x_0^2 will be *bounded* too.
  ii. Since *one* side of each rectangle is h that is *small* and the other side of each rectangle is x_0, that is *bounded*, the area of each rectangle will be *small*.
  iii. Since the *two* sides of the little square are h, that is *small*, the area of the little square is going to be h^2, that is even *smaller* than the area of each rectangle.
- Is more **efficient** because we can get at whatever size we need while, in the computational approach, we must do the whole multiplication even

addition formula for cubes

if all we want is the first or the second term.

- Will help us later deal economically with $(x_0+h)^3$, $(x_0+h)^4$ etc whereas, not only will the length of the *computational approach* get very rapidly out of hand, but, as we shall see, since we will never need more than the first two or three terms of the result, the computational approach will also become more and more inefficient.

In any case, we have

THEOREM 23.1 Addition Formula For Squares .

$$(x_0 + h)^2 = x_0^2 + 2x_0h + h^2$$

23.2 Dimension $n = 3$: $(x_0 + h)^3$ (Cubes)

In order to get the *local input-output rule* of a given cubic function near a given *bounded input* x_0, we need first to get an *addition formula* to give us $(x_0 + h)^3$ in terms of x_0 and h and which we will call the **addition formula for cubes**. We prefer to establish it here, ahead of time, rather than in the midst of the developing the local input-output rule. There are two approaches.

1. The *computational approach*. We "just" multiply three copies of (x_0+h):

a. We begin by multiplying two copies of $(x_0 + h)$:

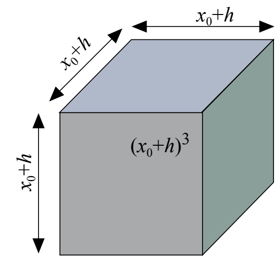
$$\begin{array}{ccccc} & & x_0 & + & h \\ & & x_0 & + & h \\ \hline & & x_0h & + & h^2 \\ x_0^2 & + & x_0h & & \\ \hline x_0^2 & + & 2x_0h & + & h^2 \end{array}$$

b. We multiply $x_0^2 + 2x_0h + h^2$, the result of the multiplication of two copies of $(x_0 + h)$, by a third copy of $(x_0 + h)$:

$$\begin{array}{ccccccc} & & x_0^2 & + & 2x_0h & + & h^2 \\ & & & & x_0 & + & h \\ \hline & & x_0^2h & + & 2x_0h^2 & + & h^3 \\ x_0^3 & + & 2x_0^2h & + & x_0h^2 & & \\ \hline x_0^3 & + & 3x_0^2h & + & 3x_0h^2 & + & h^3 \end{array}$$

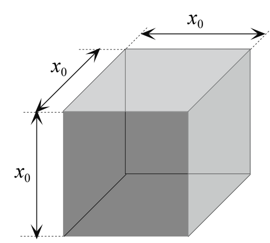
2. The *graphic approach*.

We go back to the real world and to the *definition of double multiplication* in terms of the *volume of a rectangle* so that $(x_0+h)^3$ is the volume of a x_0+h by x_0+h by x_0+h cube:

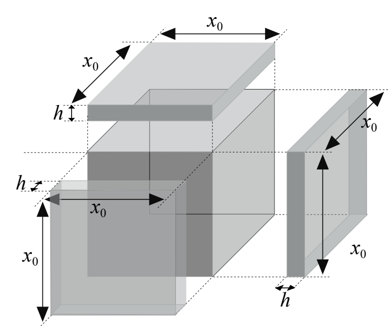


What we then do is to get this volume by starting with an x_0 by x_0 by x_0 cube, the initial cube, and then augment the sides of the initial cube by h and see how the volume increases. For the sake of clarity, we will augment the cube one step at a time:

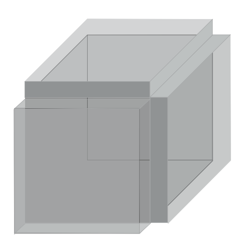
i. The sides of the initital cube are equal to x_0 and the volume of the initial cube is therefore x_0^3:



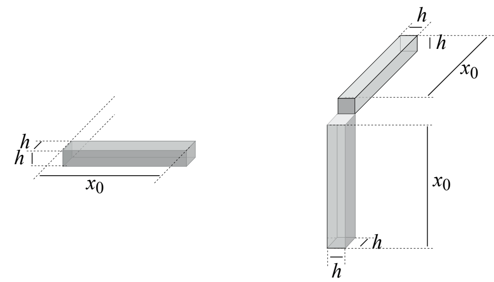
ii. We now augment the sides of the initial cube by h in each dimension which adds three x_0+h by x_0+h by h **slabs** to the initial cube:



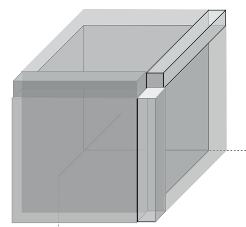
iii. We glue the three *slabs* to the *initial cube* which, however, leaves three **grooves**:



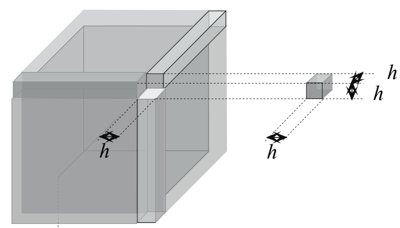
iv. We fill the three *grooves* with three x_0 by h by h **rods**:



v. We glue the three *rods* in the three *grooves* which leaves a hole in the corner:

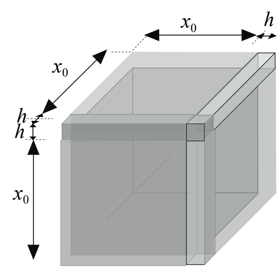


vi. We complete the enlargement of the *initial cube* by adding one h by h by h cube to fill the *indentation* in the corner:



vii. Altogether, the volume of the *enlarged cube*, whose side is $x_0 + h$ is therefore:

$$x_0^3 + 3x_0^2h + 3x_0h + h^2 + h^3$$



3. Contrary to what one might think at first, the *graphic approach* is greatly preferable because it has three major advantages over the *computational approach*:

i. The terms in the sum automatically come in order of *diminishing sizes*. Indeed, since x_0 is *bounded* and h is *small*,

- all three dimensions of the "initial cube" are *bounded*, say "in the ones", so x_0^3 is also *bounded* or "in the ones",
- two dimensions of the "slabs" are *bounded*, say "in the ones", but the third dimension is *small*, say "in the tenths", so the volume of the slabs is *small*: since x_0 is "in the ones" and h is "in the tenths", then $3x_0^2h$ will also be "in the tenths",

pattern

- one dimension of the "rods" is bounded, say "in the ones", but the other two dimensions are *small*, say "in the tenths", so the volume of the rods is *smaller*: since x_0 is "in the ones" and h is "in the tenths", then $3x_0h^2$ will be "in the hunfredths",
- all three dimensions of the "little cube" are *small* so that the volume of the little cube is even smaller than the volume of the slabs: if h is "in the tenths", then h^3 will be "in the thousandths".

ii. If all we need is only a particular one of the terms, and this will very often be the case, we can get it straight from the picture without having to go through the whole multiplication.
iii. Later on, when we shall need formulas for $(x_0 + h)^4$, $(x_0 + h)^5$, etc, not only will the length of the *computational approach* get very rapidly out of hand but, as we shall see, since we will never need more than the first two or three terms of the result, the computational approach will also become more and more inefficient. On the other hand, even though we will not be able to draw pictures as we have been able to do so far, one can extend the patterns we have found so far in the *graphic approach* and the graphic approach will thus survive.

In any case, we have

THEOREM 23.2 Addition Formula For Cubes .

$$(x_0 + h)^3 = x_0^3 + 3x_0^2h + 3x_0h^2 + h^3$$

23.3 Dimension $n \geqq 4$: $(x_0 + h)^n$ (Binomial Theorem)

Since rational functions can involve polynomial functions of any degree, we need to investigate *Addition Formulas* for all power functions with non-negative exponents.

1. In order to find a *general procedure* for getting the addition formulas for $(x_0+h)^n$ when the exponent n no matter what n, we look for a **pattern** from the addition formulas that we already have by writing all that normally "goes without saying":
For the *exceptional* exponents 0 and +1 we have

$$(x_0 + h)^0 = \boxed{1} \cdot x_0^{0}h^{0}$$

$$(x_0 + h)^1 = \boxed{1} \cdot x_0^{1}h^{0} + \boxed{1} \cdot x_0^{0}h^{1}$$

For the *regular* exponent 2 , we saw that the *addition formula* is

$$(x_0 + h)^2 = x_0^2 + 2x_0 h + h^2$$

that is

$$(x_0 + h)^2 = \boxed{1} \cdot x_0^{\boxed{2}} h^{\boxed{0}} + \boxed{2} \cdot x_0^{\boxed{1}} h^{\boxed{1}} + \boxed{1} \cdot x_0^{\boxed{0}} h^{\boxed{2}}$$

For the *regular* exponent 3, we saw that the *addition formula* is

$$(x_0 + h)^3 = x_0^3 + 3x_0^2 h + 3x_0 h^2 + h^3$$

that is

$$(x_0 + h)^3 = \boxed{1} \cdot x_0^{\boxed{3}} h^{\boxed{0}} + \boxed{3} \cdot x_0^{\boxed{2}} h^{\boxed{1}} + \boxed{3} \cdot x_0^{\boxed{1}} h^{\boxed{2}} + \boxed{1} \cdot x_0^{\boxed{0}} h^{\boxed{3}}$$

2. i. In other words, the procedure for finding the *powers* would seem to be:

i. We make 3 rows of 2 copies of x_0 with multiplication signs in-between:

$$\begin{array}{c} x_0 \cdot x_0 \\ x_0 \cdot x_0 \\ x_0 \cdot x_0 \end{array}$$

ii. On row 1, we keep all the copies of x_0: $x_0 \cdot x_0$
On row 2, we replace one copy of x_0 by a copy of h $x_0 \cdot \boxed{h}$
On row 3, we replace one more copy of x_0 by a copy of h $\boxed{h} \cdot \boxed{h}$

==========OK SO FAR==========

$$\begin{array}{c} \boxed{x_0 \bullet x_0 \bullet x_0} \\ \boxed{x_0 \bullet x_0 \bullet x_0} \\ \boxed{x_0 \bullet x_0 \bullet x_0} \end{array}$$

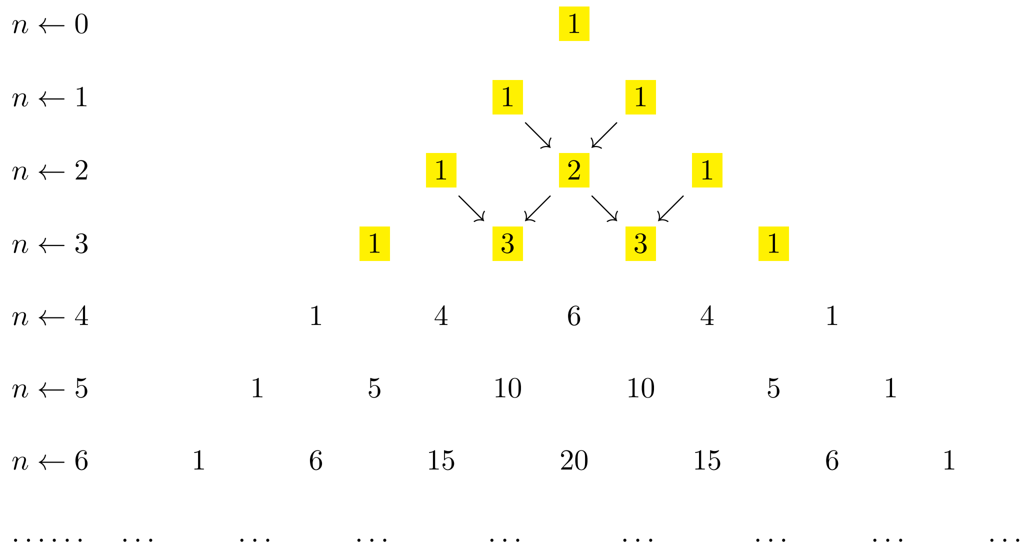
iii. Starting with the first of the 3 rows that we tacked-on in step **ii.** and going down, we replace in each row, starting from the right, one more copy of x_0 by a copy of h than in the preceding row:

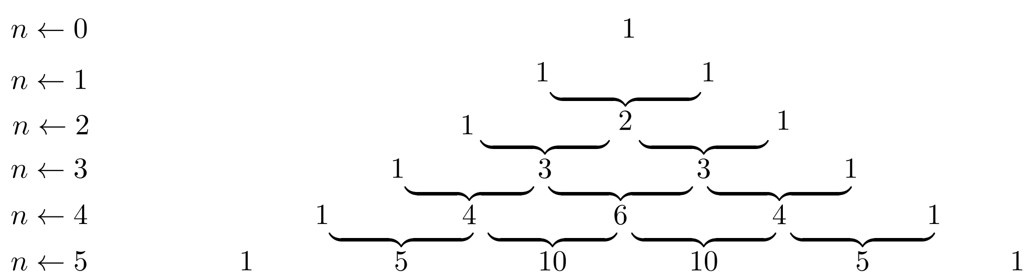
$$\begin{array}{c} x_0 \bullet x_0 \bullet x_0 \\ x_0 \bullet x_0 \bullet \boxed{h} \\ x_0 \bullet \boxed{h \bullet h} \\ \boxed{h \bullet h \bullet h} \end{array}$$

ii. The addition formula for exponent 3 becomes

$$(x_0 + h)^3 = \boxed{1} \cdot x_0^{\boxed{3}} h^{\boxed{0}} + \boxed{3} \cdot x_0^{\boxed{2}} h^{\boxed{1}} + \boxed{3} \cdot x_0^{\boxed{1}} h^{\boxed{2}} + \boxed{1} \cdot x_0^{\boxed{0}} h^{\boxed{3}}$$

3. For the coefficients, we will use **PASCAL'S TRIANGLE**:


n ← 0 1
n ← 1 1 1
n ← 2 1 2 1
n ← 3 1 3 3 1
n ← 4 1 4 6 4 1
n ← 5 1 5 10 10 5 1
n ← 6 1 6 15 20 15 6 1
....


n ← 0 1
n ← 1 1 1
n ← 2 1 2 1
n ← 3 1 3 3 1
n ← 4 1 4 6 4 1
n ← 5 1 5 10 10 5 1

4. So, in order to find a *general procedure* for getting the addition formulas for $(x_0+h)^n$ when the exponent n is higher than 3, we will first try to look for a **pattern** from the above two addition formulas by writing all that "goes without saying": pattern

i. The addition formula for exponent 2 becomes

$$(x_0+h)^2 = \boxed{1}\cdot x_0^{\boxed{2}}h^{\boxed{0}} + \boxed{2}\cdot x_0^{\boxed{1}}h^{\boxed{1}} + \boxed{1}\cdot x_0^{\boxed{0}}h^{\boxed{2}}$$

In other words, the procedure for finding the *powers* would seem to be:

i. We make 3 rows of 2 copies of x_0 with multiplication signs in-between:

$$\begin{array}{c} x_0\cdot x_0 \\ x_0\cdot x_0 \\ x_0\cdot x_0 \end{array}$$

ii. On row 1, we keep all the copies of x_0: $x_0\cdot x_0$
On row 2, we replace one copy of x_0 by a copy of h $x_0\cdot \boxed{h}$
On row 3, we replace one more copy of x_0 by a copy of h $\boxed{h}\cdot\boxed{h}$

=========OK SO FAR=========

$$\begin{array}{c} \boxed{x_0 \bullet x_0 \bullet x_0} \\ \boxed{x_0 \bullet x_0 \bullet x_0} \\ \boxed{x_0 \bullet x_0 \bullet x_0} \end{array}$$

iii. Starting with the first of the 3 rows that we tacked-on in step **ii.** and going down, we replace in each row, starting from the right, one more copy of x_0 by a copy of h than in the preceding row:

$$\begin{array}{c} x_0 \bullet x_0 \bullet x_0 \\ x_0 \bullet x_0 \bullet \boxed{h} \\ x_0 \bullet \boxed{h \bullet h} \\ \boxed{h \bullet h \bullet h} \end{array}$$

ii. The addition formula for exponent 3 becomes

$$(x_0+h)^3 = \boxed{1}\cdot x_0^{\boxed{3}}h^{\boxed{0}} + \boxed{3}\cdot x_0^{\boxed{2}}h^{\boxed{1}} + \boxed{3}\cdot x_0^{\boxed{1}}h^{\boxed{2}} + \boxed{1}\cdot x_0^{\boxed{0}}h^{\boxed{3}}$$

Looking separately at the *powers* and at the *coefficients* in the addition formulas

=======Begin WORK ZONE=======

without letting anything "go without saying" and see what *particular procedure* would give each one of them.

a. When the *exponent* is 3, we have:

$$(x_0+h)^3 = x_0^3 + 3x_0^2h + 3x_0h^2 + h^3$$

that is, writing in all that "goes without saying",

$$= 1x_0^{+3}h^0 + 3x_0^{+2}h^{+1} + 3x_0^{+1}h^{+2} + 1x_0^{0}h^{+3}$$

that is, replacing the short hand by the long hand

$$\begin{aligned} = \quad & 1 \bullet x_0 \bullet x_0 \bullet x_0 \\ & + 3 \bullet x_0 \bullet x_0 \bullet h \\ & + 3 \bullet x_0 \bullet h \bullet h \\ & + 1 \bullet h \bullet h \bullet h \end{aligned}$$

So, looking separately at the *powers* and at the *coefficients* in the addition formula for $(x_0 + h)^3$:

- The procedure for finding the *powers* would seem to be:

  i. We make a row of 3 copies of x_0 with multiplication signs in-between:

  $$x_0 \bullet x_0 \bullet x_0$$

  ii. We tack on 3 copies of the row we wrote in step **i.**

  $$\begin{array}{c} x_0 \bullet x_0 \bullet x_0 \\ x_0 \bullet x_0 \bullet x_0 \\ x_0 \bullet x_0 \bullet x_0 \\ x_0 \bullet x_0 \bullet x_0 \end{array}$$

  iii. Starting with the first of the 3 rows that we tacked-on in step **ii.** and going down, we replace in each row, starting from the right, one more copy of x_0 by a copy of h than in the preceding row:

  $$\begin{array}{c} x_0 \bullet x_0 \bullet x_0 \\ x_0 \bullet x_0 \bullet h \\ x_0 \bullet h \bullet h \\ h \bullet h \bullet h \end{array}$$

- The procedure for finding the *coefficients*

  $$1 \qquad 3 \qquad 3 \qquad 1$$

  is certainly not obvious and we leave it alone for the time being.

b. When the exponent is 2, we have

$$(x_0 + h)^2 = x_0^2 + 2x_0h + h^2$$

that is, writing in all that "goes without saying",

$$= 1x_0^2h^0 + 2x_0^1h^1 + 1x_0^0h^2$$

that is, replacing the short hand by the long hand

$$\begin{aligned} = \quad & 1 \bullet x_0 \bullet x_0 \\ & + 2 \bullet x_0 \bullet h \\ & + 1 \bullet h \bullet h \end{aligned}$$

So, looking separately at the *powers* and at the *coefficients* in the addition formula for $(x_0 + h)^2$:

- The procedure for finding the *powers* would seem to be the same as for the previous addition formula:

  i. We make a row of 2 copies of x_0 with multiplication signs in-between:

  $$x_0 \bullet x_0$$

  ii. We tack on 2 copies of the row we wrote in step **i.**

  $$\begin{array}{c} x_0 \bullet x_0 \\ x_0 \bullet x_0 \\ x_0 \bullet x_0 \end{array}$$

  iii. Starting with the first of the 2 rows that we tacked-on in step **ii.** and going down, we replace in each row, starting from the right, one more copy of x_0 by a copy of h than in the preceding row:

  $$\begin{array}{c} x_0 \bullet x_0 \\ x_0 \bullet h \\ h \bullet h \end{array}$$

- The procedure for finding the *coefficients*

  $$1 \qquad 2 \qquad 1$$

  is certainly not obvious and we leave it alone for the time being.

c. Finally, for the sake of completion, we look at what happens when the exponent is 1. We have

$$(x_0 + h)^1 = x_0 + h$$

that is, writing in all that "goes without saying",

$$= 1x_0^1h^0 + 1x_0^0h^1$$

that is, replacing the short hand by the long hand

$$\begin{aligned} = \quad & 1 \bullet x_0 \\ & + 1 \bullet h \end{aligned}$$

So, looking separately at the *powers* and at the *coefficients* in the "addition formula" for $(x_0 + h)^1$:

- The procedure for finding the *powers* would seem to be the same as for the previous addition formula:

  i. We make a row of 1 copy of x_0 with no room for multiplication signs in-between:

  $$x_0$$

  ii. We tack on 1 copy of the row we wrote in step **i.**

  $$\begin{array}{c} x_0 \\ x_0 \end{array}$$

PASCAL TRIANGLE
parent-entries

iii. Starting with the first–and only– row that we wrote in step **ii.**, we replace in that row, the one copy of x_0 by a copy of h:

x_0
h

- The procedure for finding the *coefficients*

  1 1

  is certainly not obvious and we leave it alone for the time being.

=======End WORK ZONE=======

5. Putting everything together, the general procedure for constructing the addition formula for $(x_0 + h)^n$ would seem to be,

- The procedure for finding the *powers* seems to be in all cases:

  i. We make a row of n copies of x_0 with multiplication signs in-between:

  ii. We tack on n copies of the row we got in step **i.**

  iii. Starting with the first of the n rows that we wrote in step **ii.** and going down, we replace in each row, starting from the right, one more copy of x_0 by a copy of h than in the preceding row:
- So far, the procedures for the *coefficients* were not obvious but when we write them starting with exponent 1 and ending with exponent 3

$n \leftarrow 1$			1		1		
$n \leftarrow 2$		1		2		1	
$n \leftarrow 3$	1		3		3		1

we can see that we can get each entry in what is called the **PASCAL TRIANGLE** by adding its two **parent-entries**, that is the two entries just above it.

EXAMPLE 23.1. The next line in the **PASCAL TRIANGLE** for $n \leftarrow 4$ would be:

$n \leftarrow 3$		1		3		3		1	
		↘	↙	↘	↙	↘	↙		
$n \leftarrow 4$	1		4		6		4		1

We thus have two procedures, one for getting the *powers* and the other for getting the *coefficients* and all that remains is to put them together.

EXAMPLE 23.2. In order to construct the addition formula for $(x_0+h)^5$, we proceed as follows:

i. We make a row of 5 copies of x_0 with multiplication signs in-between:

$x_0 \bullet x_0 \bullet x_0 \bullet x_0 \bullet x_0$

ii. We tack on 5 copies of the row we wrote in step **i.**

$$\begin{array}{c} x_0 \bullet x_0 \bullet x_0 \bullet x_0 \bullet x_0 \\ x_0 \bullet x_0 \bullet x_0 \bullet x_0 \bullet x_0 \\ x_0 \bullet x_0 \bullet x_0 \bullet x_0 \bullet x_0 \\ x_0 \bullet x_0 \bullet x_0 \bullet x_0 \bullet x_0 \\ x_0 \bullet x_0 \bullet x_0 \bullet x_0 \bullet x_0 \\ x_0 \bullet x_0 \bullet x_0 \bullet x_0 \bullet x_0 \end{array}$$

iii. Starting with the first of the 5 rows that we tacked-on in step **ii.** and going down, we replace in each row, starting from the right, one more copy of x_0 by a copy of h than in the preceding row:

$$\begin{array}{c} x_0 \bullet x_0 \bullet x_0 \bullet x_0 \bullet x_0 \\ x_0 \bullet x_0 \bullet x_0 \bullet x_0 \bullet h \\ x_0 \bullet x_0 \bullet x_0 \bullet h \bullet h \\ x_0 \bullet h \bullet h \bullet h \bullet h \\ x_0 \bullet h \bullet h \bullet h \bullet h \\ h \bullet h \bullet h \bullet h \bullet h \end{array}$$

iv. We get the coefficients from the **PASCAL TRIANGLE**:

$$\begin{array}{lccccccccccc} n \leftarrow 4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\ & \swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & & \\ n \leftarrow 5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

6. *Proving* that all this is indeed the case would involve more work than we are willing to do here and so we will take the following for granted:

THEOREM 23.3 BINOMIAL THEOREM . The addition formula for a binomial of degree n is:

$$\begin{aligned} (x_0 + h)^n = {} & \frac{1}{1} x_0^n h^0 \\ & + \frac{n}{1} x_0^{n-1} h^{+1} \\ & + \frac{n(n-1)}{1 \cdot 2} x_0^{n-2} h^{+2} \\ & + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x_0^{n-3} h^{+3} \\ & + \ \dots \\ & + \frac{n(n-1)(n-2)\cdots(1)}{1 \cdot 2 \cdot 3 \cdots n} x_0^0 h^n \end{aligned}$$

where it is interesting to note that, no matter what n,

$$\frac{n(n-1)(n-2)\cdots(1)}{1 \cdot 2 \cdot 3 \cdots n} = 1$$

because it corresponds to the fact that the rows in the **PASCAL TRIANGLE** are symmetrical.

7. According to the **BINOMIAL THEOREM**, when $n = 0$ we should slanted␣row

have

$$(x_0 + h)^0 = x_0^0 h^0$$

and since the coefficient 1 goes without saying:

$$= 1x_0^0 h^0 = 1$$

This is of course as it should be but what this says is that the “zeroth” line in the **PASCAL TRIANGLE** is 1 so that the “complete” **PASCAL TRIANGLE** is:

$n \leftarrow 0$ 1

$n \leftarrow 1$ 1 1

$n \leftarrow 2$ 1 2 1

$n \leftarrow 3$ 1 3 3 1

$n \leftarrow 4$ 1 4 6 4 1

$n \leftarrow 5$ 1 5 10 10 5 1

$n \leftarrow 6$ 1 6 15 20 15 6 1

....

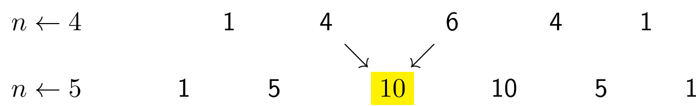
8. Usually, though, we do not need the *complete addition formulas* but only the first three or, at worst, the first four terms and so using the **PASCAL TRIANGLE** as we did above would turn out to be a huge waste of energy.

The key to a more economical way of using the **PASCAL TRIANGLE** is to look at the **slanted rows** and notice that:

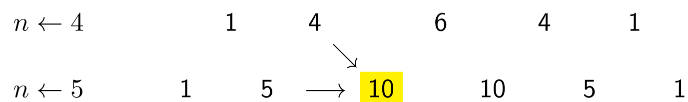
- The numbers in the *first* slanted row are all 1 and they are the coefficients of the h^0 powers in the addition formulas for $(x_0+h)^1$, $(x_0+h)^2$, $(x_0+h)^3$, $(x_0+h)^4$, $(x_0+h)^5$, $(x_0+h)^6$, etc, which corresponds to the fact that the coefficient of the h^0 power in x_0^n is 1 no matter what n.
- The numbers in the *second* slanted row are $1, 2, 3, 4, 5, 6, \ldots$ and they are the coefficients of the h^{+1} powers in the addition formulas for $(x_0+h)^1$, $(x_0+h)^2$, $(x_0+h)^3$, $(x_0+h)^4$, $(x_0+h)^5$, $(x_0+h)^6$, etc, which shows that the coefficient of the h^{+1} power in x_0^n is n no matter what n.
- We check that the *third* slanted row, $1, 3, 6, 10, 15, \ldots$ are the coefficients of the h^{+2} powers in the addition formulas for $(x_0+h)^1$, $(x_0+h)^2$,

$(x_0 + h)^3$, $(x_0 + h)^4$, $(x_0 + h)^5$, $(x_0 + h)^6$, etc, and we observe that the coefficient of the h^{+2} power in x_0^n is the half-product of the two coefficients to its left and therefore equal to $\frac{n(n-1)}{2}$ no matter what n.

EXAMPLE 23.3. Instead of getting the coefficient of h^{+2} in the addition formula for $(x_0 + h)^5$ as the sum of its two "parents", 4 and 6:

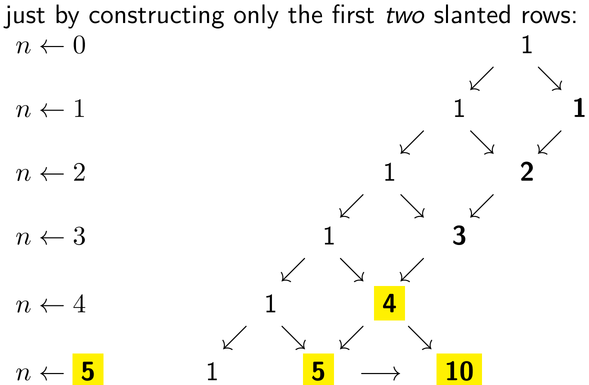


we can get the coefficient of h^{+2} in the addition formula for $(x_0 + h)^5$ as the half-product of the two numbers to its left, 5 and 4:



As a result, we only need, and can construct only, the first two slanted rows of the **PASCAL TRIANGLE**.

EXAMPLE 23.4. In the addition formula for exponent 5, we get the first *three* coefficients just by constructing only the first *two* slanted rows:



where 10 is the half-product of the second entries in the last two horizontal rows.

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