cycle
step

Chapter 24

Polynomial Divisions

24.1 Division in Descending Exponents

Since *decimal numbers* are combinations of powers of TEN, it should not be surprising that the procedure for dividing decimal numbers should also work for *polynomials* which are combinations of powers of x.

The *procedure* consists of successive **cycles**, one for each monomial in the quotient. During each of these *cycles*, we go through four **steps**:

Step I. We find each *monomial* of the *quotient* by dividing the *first monomial* in the divisor into the *first monomial* of the previous partial remainder or, if there is not yet a partial remainder, from the *dividend*.

Step II. We find the *partial product* by multiplying the *full divisor* by the *monomial* of the quotient we found in Step I.

Step III. We find the *partial remainder* by subtracting the *partial product* we found in Step II from the previous partial remainder or, if there is not yet a partial remainder, from the *dividend*.

Step IV. We decide if we want to:

- *stop* the division,

or

- *continue* the division and go through another *cycle*.

Example 24.1. In order to compute $\dfrac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5 - 2}$, we set up the division of $-3x^2 + 5 - 2$ "into" $-12x^3 + 11x^2 - 17x + 1$

$$-3x^2 \ +5x \ -2 \ \big) \ \overline{-12x^3 \quad +11x^2 \quad -17x \quad +1}$$

and we proceed as follows:

CYCLE 1. Step I. We find the *first monomial in the quotient* by dividing the *first monomial in the divisor*, $-3x^2$, into the *first monomial of the dividend*, $-12x^3$, which give us $\dfrac{-12x^3}{-3x^2} = +4x$ as first term in the quotient:

$$\begin{array}{rrrr|rrrr} & & & & +4x & & & \\ \cline{5-8} -3x^2 & +5x & -2 &) & -12x^3 & +11x^2 & -17x & +1 \end{array}$$

Step II. We find the *first partial product* by multiplying the *full divisor* by the *first monomial in the quotient*:

$$\begin{array}{rrrr|rrrr} & & & & +4x & & & \\ \cline{5-8} -3x^2 & +5x & -2 &) & -12x^3 & +11x^2 & -17x & +1 \\ \text{First partial product:} & & & & -12x^3 & +20x^2 & -8x & \end{array}$$

Step III. We find the *first partial remainder* by *subtracting* the first partial product from the full dividend:

$$\begin{array}{rrrr|rrrr} & & & & +4x & & & \\ \cline{5-8} -3x^2 & +5x & -2 &) & -12x^3 & +11x^2 & -17x & +1 \\ & & & \ominus & -12x^3 & +20x^2 & -8x & \end{array}$$

But to *subtract* the first partial product means to *add the opposite* of the first partial product to the full dividend:

$$\begin{array}{rrrr|rrrr} & & & & +4x & & & \\ \cline{5-8} -3x^2 & +5x & -2 &) & -12x^3 & +11x^2 & -17x & +1 \\ & & & \oplus & +12x^3 & -20x^2 & +8x & \\ \cline{5-8} \text{First remainder:} & & & & +0x^3 & -9x^2 & -9x & +1 \end{array}$$

Step IV. We decide if we want to *stop* or *continue* the division.

- If we decide to *stop* the division,
  - the *quotient* of the division is $+4x$.
  - the *remainder* of the division is $-9x^2 - 8x + 1$

  If we don't care about the *remainder*, we write:

  $$\frac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5x - 2} = +4x + [...]$$

  where we write $+$ [...] as a reminder that $\dfrac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5x - 2}$ is not exactly equal to $+4x$ since there was a *remainder*.
- If we decide to *continue* the division, we begin a new cycle

CYCLE 2. Step I. We find the *second monomial in the quotient* by dividing the *first* *monomial in the divisor*, $-3x^2$, into *the first monomial in the first partial remainder*, $-9x^2$, which gives us $\frac{-9x^2}{-3x^2} = +3$ for the second term of the quotient:

$$\begin{array}{rrr|rrrr} & & & +4x & +3 & & \\ \cline{4-7} -3x^2 & +5x & -2 \;) & -12x^3 & +11x^2 & -17x & +1 \\ & & & -12x^3 & +20x^2 & -8x & \\ \cline{4-7} & & & & -9x^2 & -9x & +1 \end{array}$$

Step II. We find the *second partial product* by multiplying the *full divisor* by the *second monomial in the quotient*:

$$\begin{array}{lrrr|rrrr} & & & & +4x & +3 & & \\ \cline{5-8} & -3x^2 & +5x & -2 \;) & -12x^3 & +11x^2 & -17x & +1 \\ & & & & -12x^3 & +20x^2 & -8x & \\ \cline{5-8} & & & & & -9x^2 & -9x & +1 \\ \text{Second partial product:} & & & & & -9x^2 & +15x & -6 \end{array}$$

Step III. We find the *second partial remainder* by *subtracting* the second partial product from the first partial remainder:

$$\begin{array}{rrr|rrrr} & & & +4x & +3 & & \\ \cline{4-7} -3x^2 & +5x & -2 \;) & -12x^3 & +11x^2 & -17x & +1 \\ & & & -12x^3 & +20x^2 & -8x & \\ \cline{4-7} & & & & -9x^2 & -9x & +1 \\ & & & \ominus & -9x^2 & +15x & -6 \end{array}$$

But to *subtract* the second partial product means to *add the opposite* of the second partial product to the first partial remainder:

$$\begin{array}{lrrr|rrrr} & & & & +4x & +3 & & \\ \cline{5-8} & -3x^2 & +5x & -2 \;) & -12x^3 & +11x^2 & -17x & +1 \\ & & & & -12x^3 & +20x^2 & -8x & \\ \cline{5-8} & & & & & -9x^2 & -9x & +1 \\ & & & & \oplus & +9x^2 & -15x & +6 \\ \cline{6-8} \text{Second remainder:} & & & & & +0x^2 & -24x & +7 \end{array}$$

Step IV. We decide if we want to *stop* or *continue* the division.

- If we decide to *stop* the division,
  - the *quotient* of the division is $+4x+3$.
  - the *remainder* of the division is $-24x+7$

  If we don't care about the *remainder*, we write:

  $$\frac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5x - 2} = +4x + 3 + [...]$$

  where we write $+$ [...] as a reminder that $\frac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5x - 2}$ is not exactly equal to $+4x+3$ since there was a *remainder*.
- If we decide to *continue* the division, we begin a new cycle

CYCLE 3. Step I. We find the *third monomial in the quotient* by dividing the *first* *monomial in the divisor*, $-3x^2$, into *the first monomial in the second partial remainder*, $-24x$ that is $\dfrac{-24x}{-3x^2} = +8x^{-1}$

$$\begin{array}{rrrrrrr}
 & & +4x & +3 & +8x^{-1} & & \\
-3x^2 & +5x & -2 \,) & -12x^3 & +11x^2 & -17x & +1 \\
 & & & -12x^3 & +20x^2 & -8x & \\
 & & & & -9x^2 & -9x & +1 \\
 & & & & +9x^2 & -15x & +6 \\
 & & & & & -24x & +7
\end{array}$$

Step II. We find the *third partial product* by multiplying the *full divisor* by the *third monomial in the quotient*:

$$\begin{array}{rrrrrrr}
 & & +4x & +3 & +8x^{-1} & & \\
-3x^2 +5x -2 \,) & -12x^3 & +11x^2 & -17x & +1 & & \\
 & -12x^3 & +20x^2 & -8x & & & \\
 & & -9x^2 & -9x & +1 & & \\
 & & +9x^2 & -15x & +6 & & \\
 & & & -24x & +7 & & \\
\text{Third partial product:} & & & -24x & +40 & -16x^{-1} &
\end{array}$$

Step III. We find the *third partial remainder* by *subtracting* the third partial product from the first partial remainder:

$$\begin{array}{rrrrrr}
 & +4x & +3 & +8x^{-1} & & \\
-3x^2 +5x -2 \,) & -12x^3 & +11x^2 & -17x & +1 & \\
 & -12x^3 & +20x^2 & -8x & & \\
 & & -9x^2 & -9x & +1 & \\
 & & +9x^2 & -15x & +6 & \\
 & & & -24x & +7 & \\
 & \ominus & & -24x & +40 & -16x^{-1}
\end{array}$$

But to *subtract* the second partial product means to *add the opposite* of the second partial product to the first partial remainder:

$$\begin{array}{rrrrrr}
 & +4x & +3 & +8x^{-1} & & \\
-3x^2 +5x -2 \,) & -12x^3 & +11x^2 & -17x & +1 & \\
 & -12x^3 & +20x^2 & -8x & & \\
 & & -9x^2 & -9x & +1 & \\
 & & +9x^2 & -15x & +6 & \\
 & & & -24x & +7 & \\
 & \oplus & & +24x & -40 & +16x^{-1} \\
\text{Third remainder:} & & & 0x & -33 & +16x^{-1}
\end{array}$$

Step IV. We decide if we want to *stop* or *continue* the division.

- If we decide to *stop* the division,
  - the *quotient* of the division is $+4x + 3 + 8x^{-1}$.
  - the *remainder* of the division is $-33 + 16x^{-1}$

  If we don't care about the *remainder*, we write:

$$\frac{-12x^3+11x^2-17x+1}{-3x^2+5x-2} = +4x+3+8x^{-1}+[...]$$

where we write $+$ [...] as a reminder that $\dfrac{-12x^3+11x^2-17x+1}{-3x^2+5x-2}$ is not exactly equal to $+4x+3+8x^{-1}$ since there was a *remainder*.

- If we decide to *continue* the division, we begin a new cycle

Just as, in arithmetic, we need not stop a division when the quotient reaches a monomial with exponent 0 because we can always divide a monomial into another and we can have *negative* exponents. In fact, again just as in arithmetic, there are cases where we absolutely need to go beyond the exponent 0 and use negative exponents. (See Epilogue.)

EXAMPLE 24.2. In order to divide $-12x^3+11x^2-17x+4$ by $-3x^2+5x-2$ down to a x^{-2} monomial, we write (in the *anglo-saxon* tradition):

$$\begin{array}{r|rrrrrr}
 & +4x & +3 & +8x^{-1} & +10x^{-2} & & \\
\hline
-3x^2 \ +5x \ -2 & -12x^3 & +11x^2 & -17x & +4 & & \\
 & +12x^3 & -20x^2 & +8x & & & \\
\cline{2-4}
 & & -9x^2 & -9x & & & \\
 & & +9x^2 & -15x & +6 & & \\
\cline{3-5}
 & & & -24x & +10 & & \\
 & & & +24x & -40 & +16x^{-1} & \\
\cline{4-6}
 & & & & -30 & -16x^{-1} & \\
 & & & & +30 & -50x^{-1} & +20x^{-2} \\
\cline{5-7}
 & & & & & -66x^{-1} & +20x^{-2}
\end{array}$$

The quotient is

$$+4x+3+8x^{-1}+10x^{-2}$$

and the remainder is

$$-66x^{-1}+20x^{-1}$$

24.2 Default Rules for Division

Since mathematicians are lazy,

- mathematicians do not write the $+$ sign in front of the coefficients of leading monomials,
- mathematicians do not write monomials with 0 coefficient,

and, most dangerously,

- mathematicians want to write only one stage in Step III but there are two traditions concerning what then to write, as a result, in Step II:
  - In the *latin* tradition, in Step II, we write the *partial product*, that is what we get it from the *multiplication*, and so in Step III, when it comes to subtracting, we visualize the *opposite of the partial product* we wrote in Step II and we oplus what we *visualize*. The advantage is that each line is exactly what we get from the previous operation.

EXAMPLE 24.3.

$$\begin{array}{r|rrrr} & +4x & +3 & & \\ \hline -3x^2\ +5x\ -2\ ) & -12x^3 & +11x^2 & -16x & +1 \\ & -12x^3 & +20x^2 & -8x & \\ \hline & & -9x^2 & -8x & +1 \end{array}$$

  - In the *anglo-saxon* tradition, we anticipate the subtraction to be done in Step III and in Step II *we write the opposite of the partial product* so in Step III we oplus what we *wrote* in Step II.

EXAMPLE 24.4.

$$\begin{array}{r|rrrr} & +4x & +3 & & \\ \hline -3x^2\ +5x\ -2\ ) & -12x^3 & +11x^2 & -16x & +1 \\ & +12x^3 & -20x^2 & +8x & \\ \hline & & -9x^2 & -8x & +1 \end{array}$$

From now on we will of course follow the *anglo-saxon* tradition.

EXAMPLE 24.5. In order to compute $\dfrac{6x^3+13x^2+13x+7}{2x+1}$, we divide $2x+1$ into $6x^3+13x^2+13x+7$:

CYCLE 1. Step I. We find the *first monomial* in the quotient by *short division*:

$$\begin{array}{r|l} & \quad 3x^2 \\ \hline 2x+1) & 6x^3+13x^2+13x+7 \end{array}$$

Step II. We get the *first opposite product* by writing the opposite of the result of the *full multiplication*

$$\begin{array}{r|l} & \quad 3x^2 \\ \hline 2x+1) & 6x^3+13x^2+13x+7 \\ & -6x^3\ -3x^2 \\ \hline \end{array}$$

Step III. We get the *first remainder* by oplussing the first opposite product

$$\begin{array}{r|l} & \quad 3x^2 \\ \hline 2x+1) & 6x^3+13x^2+13x+7 \\ & -6x^3\ -3x^2 \\ \hline & \qquad 10x^2+13x \end{array}$$

Step IV. We decide if we want to stop or continue the division

- If we decide to *stop* the division,
  - ∗ the *quotient* of the division is $+3x^{+2}$.

* the *remainder* of the division is $+10x^{+2} + 13x$

If we don't care about the *remainder*, we write:

$$\frac{6x^3 + 13x^2 + 13x + 7}{2x + 1} = +3x^{+2} + (\ldots)$$

where we write $+ (\ldots)$ as a reminder that $\frac{6x^3 + 13x^2 + 13x + 7}{2x + 1}$ is not exactly equal to $+3x^{+2}$ since there was a *remainder*.

– If we decide to *continue* the division, we begin a new cycle

CYCLE 2. Step I. We find the *second monomial* in the quotient by *short division*:

$$\begin{array}{r|l} & \quad\; 3x^2 \quad + 5x \\ \hline 2x + 1) & 6x^3 + 13x^2 + 13x + 7 \\ & -6x^3 \; - 3x^2 \\ \hline & \qquad\; 10x^2 + 13x \end{array}$$

Step II. We get the *second opposite product* by writing the opposite of the result of the *full multiplication*

$$\begin{array}{r|l} & \quad\; 3x^2 \quad + 5x \\ \hline 2x + 1) & 6x^3 + 13x^2 + 13x + 7 \\ & -6x^3 \; - 3x^2 \\ \hline & \qquad\; 10x^2 + 13x \\ & \qquad -10x^2 \; - 5x \\ \hline \end{array}$$

Step III. We get the *second remainder* by oplussing the fir second st opposite product

$$\begin{array}{r|l} & \quad\; 3x^2 \quad + 5x \\ \hline 2x + 1) & 6x^3 + 13x^2 + 13x + 7 \\ & -6x^3 \; - 3x^2 \\ \hline & \qquad\; 10x^2 + 13x \\ & \qquad -10x^2 \; - 5x \\ \hline & \qquad\qquad\qquad 8x + 7 \end{array}$$

Step IV. We decide if we want to stop or continue the division

– If we decide to *stop* the division,

* the *quotient* of the division is $+3x^{+2} + 5x$.
* the *remainder* of the division is $+8x + 7$

If we don't care about the *remainder*, we write:

$$\frac{6x^3 + 13x^2 + 13x + 7}{2x + 1} = +3x^{+2} + 5x + (\ldots)$$

where we write $+ (\ldots)$ as a reminder that $\frac{6x^3 + 13x^2 + 13x + 7}{2x + 1}$ is not exactly equal to $+3x^{+2} + 5x$ since there was a *remainder*.

– If we decide to *continue* the division, we begin a new cycle

CYCLE 3. Step I. We find the *third monomial* in the quotient by *short division*:

$$
\begin{array}{r|l}
 & 3x^2 \quad + 5x + 4 \\ \hline
2x+1 & 6x^3 + 13x^2 + 13x + 7 \\
 & -6x^3 \quad - 3x^2 \\ \cline{2-2}
 & 10x^2 + 13x \\
 & -10x^2 \quad - 5x \\ \cline{2-2}
 & 8x + 7
\end{array}
$$

Step II. We get the *third opposite product* by writing the opposite of the result of the *full multiplication*

$$
\begin{array}{r|l}
 & 3x^2 \quad + 5x + 4 \\ \hline
2x+1 & 6x^3 + 13x^2 + 13x + 7 \\
 & -6x^3 \quad - 3x^2 \\ \cline{2-2}
 & 10x^2 + 13x \\
 & -10x^2 \quad - 5x \\ \cline{2-2}
 & 8x + 7 \\
 & -8x - 4 \\ \cline{2-2}
\end{array}
$$

Step III. We get the *third remainder* by oplussing the third opposite product

$$
\begin{array}{r|l}
 & 3x^2 \quad + 5x + 4 \\ \hline
2x+1 & 6x^3 + 13x^2 + 13x + 7 \\
 & -6x^3 \quad - 3x^2 \\ \cline{2-2}
 & 10x^2 + 13x \\
 & -10x^2 \quad - 5x \\ \cline{2-2}
 & 8x + 7 \\
 & -8x - 4 \\ \cline{2-2}
 & 3
\end{array}
$$

Step IV. We decide if we want to stop or continue the division

- If we decide to *stop* the division,
  * the *quotient* of the division is $+3x^{+2} + 5x + 4$.
  * the *remainder* of the division is $+3$

  If we don't care about the *remainder*, we write:

  $$\frac{6x^3 + 13x^2 + 13x + 7}{2x + 1} = +3x^{+2} + 5x + 4 + (...)$$

  where we write $+\ (...)$ as a reminder that $\dfrac{6x^3 + 13x^2 + 13x + 7}{2x + 1}$ is not exactly equal to $+3x^{+2} + 5x + 4$ since there was a *remainder*.
- If we decide to *continue* the division, we begin a new cycle

• When writing the partial remainders, we do not write the monomials beyond those that result from subtracting the *partial product*.

EXAMPLE 24.6.

	$+4x$	$+3$		
$-3x^2$ $+5x$ -2)	$-12x^3$	$+11x^2$	$-16x$	$+1$
First *opposite* partial product:	$+12x^3$	$-20x^2$	$+8x$	
First remainder:		$-9x^2$	$-8x$	
Second *opposite* partial product:		$+9x^2$	$-15x$	$+6$
Second remainder:			$-23x$	$+7$

The danger here is that, when we do the next subtraction, we may subtract from 0 rather than from the monomial that was left unwritten in the partial remainder.

24.3 Comparison With Arithmetic Division

The procedure to divide polynomials is in fact a lot simpler than the procedure for dividing in ARITHMETIC:

- There is never any "carryover"
- The first term of each partial remainder always has coefficient 0
- There are no Trials in **Step I** because, when we divide the first monomial in the divisor into the first monomial of a partial remainder, we always get a coefficient for the corresponding monomial in the quotient and the worst that can happen is that this coefficient is a fraction.

24.4 Division in Ascending Order Of Exponents

Fortunately, the procedure is exactly the same as in the case of division in descending order of exponents and so we will just look at an example.

EXAMPLE 24.7. In order to compute $\dfrac{-12+23h-h^2-2h^3}{-3+2h}$, we divide $-3+2h$ into $-12+23h-h^2-2h^3$:

	$+4$	$-5h$	$-3h^2$	
-3 $+2h$)	-12	$+23h$	$-h^2$	$-2h^3$
First *opposite* partial product:	$+12$	$-8h$		
First remainder:		$+15h$	$-h^2$	$-2h^3$
Second *opposite* partial product:		$-15h$	$+10h^2$	
Second remainder:			$+9h^2$	$-2h^3$
Third *opposite* partial product:			$-9h^2$	$+6h^3$
Third remainder:				$+4h^3$

- If we decide to *stop* the division,
  - the *quotient* of the division is $+4-5h-3h^2$.

– the *remainder* of the division is $+4h^3$. Observe that if we replace the unspecified numerator h by, say, 0.2, then the remainder is equal to $4 \bullet 0.2^3 = 4 \bullet 0.008 = 0.032$ which is indeed small.

If we don't care about the *remainder*, we write:

$$\frac{-12 + 23h - h^2 - 2h^3}{-3 + 2h} = +4 - 5h - 3h^2 + [...]$$

where we write $+$ [...] as a reminder that $\dfrac{-12 + 23h - h^2 - 2h^3}{-3 + 2h}$ is not exactly equal to $+4 - 5h - 3h^2$ since there was a *remainder*.

- If we were to decide to *continue* the division, we would begin a new cycle

EXAMPLE 24.8. In order to divide $2x^3+5x^2-6$ by $3x-1$ we write (in the *anglo-saxon* tradition):

$$
\begin{array}{rrrrr}
 & & \frac{2}{3}x^2 & +\frac{17}{9}x & +\frac{17}{27} \\
\cline{2-5}
3x-1\big) & 2x^3 & +5x^2 & & -6 \\
 & -2x^3 & +\frac{2}{3}x^2 & & \\
\cline{2-3}
 & & \frac{17}{3}x^2 & & \\
 & & -\frac{17}{3}x^2 & +\frac{17}{9}x & \\
\cline{3-4}
 & & & \frac{17}{9}x & -6 \\
 & & & -\frac{17}{9}x & +\frac{17}{27} \\
\cline{4-5}
 & & & & -\frac{145}{27}
\end{array}
$$

The *quotient* is

$$+\tfrac{2}{3}x^2 + \tfrac{17}{9}x + \tfrac{17}{27}$$

The *remainder* is

$$-\tfrac{145}{27}$$

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