base function
add-on number
add-on function
sum function

Chapter 8

Prelude To Polynomial Functions

As already mentioned, monomial functions will be the building blocks from which all the functions we will be investigating in this text are built from. So we will always have to use more than a single monomial function at a time.

8.1 Adding Functions

1. Given a function, to which we will refer as **base function**, one often needs to add a number to each output that the base function returns. Whether or not this **add-on number** remains the same regardless of the input or differs depending on the input, we can look upon the add-on number as being itself the output returned for the same input by some other function to which we will refer as **add-on function**. (If the add-on number is the same regardless of the input this just means that the add-on function is a *constant function*.)
There is then going to be a third function, to be referred as **sum function**, which, for each input, will return the output returned by the base function plus the add-on number returned by the add-on function for that input.
In other words, given the two functions

bar graph
bar

$$x \xrightarrow{BASE} BASE(x)$$

and

$$x \xrightarrow{ADD\text{-}ON} ADD\text{-}ON(x)$$

there will be a third function specified as

$$x \xrightarrow{SUM} SUM(x) = BASE(x) + ADD\text{-}ON(x)$$

2. In sciences such as BIOLOGY, PSYCHOLOGY and ECONOMICS the three functions are often in *tabular* form.

EXAMPLE 8.1. When we shop online for, say for a textbook, we first see a *price list*—the *base function*. However, a *shipping charge*, which might or might not depend on the textbook, is usually added-on to the *list price* and is given by the *Shipping charge list*—the *add-on function*. The price we end-up having to pay is thus given by the *actual price list*—the *sum function*.

x	$\xrightarrow{LIST} LIST(x)$
English	140
History	80
Biology	130
Math	10
Poetry	70

x	$\xrightarrow{SHIP} SHIP(x)$
English	13.15
History	3.45
Biology	7.25
Math	3.75
Poetry	5.32

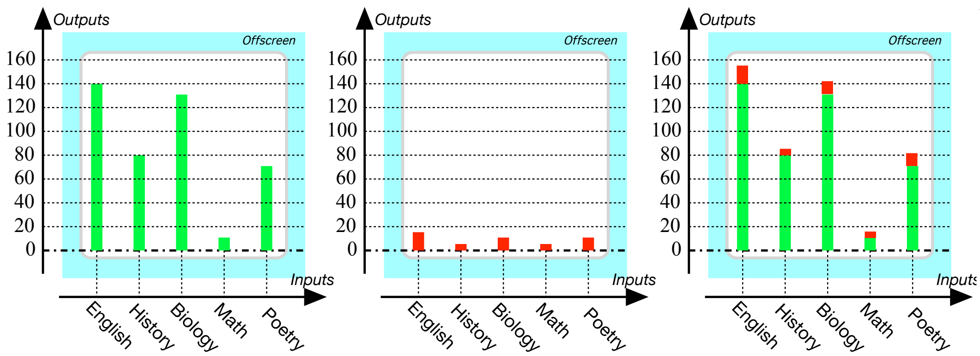
x	$\xrightarrow{PAY} PAY(x)$
English	140+13.15 = 153.15
History	80+3.45 = 83.45
Biology	130+7.35 = 137.25
Math	10+3.75 = 13.75
Poetry	70+5.32 = 75.32

which says, for instance, that while the *list price* of the English textbook is $140, a *shipping charge* of $13.15 brings the price to be *paid* to $140 + $13.15 = $153.15.

3. Instead of representing the functions by *tables*, one might want to represent them by *graphs*. Rather than to use *plots*, though, one often uses **bar graphs** in which the pieces of input level lines that are between the 0-*output level line* and the *plot point* are highlighted into **bars**.

EXAMPLE 8.2. The situation in the above example would be represented by the following bar graphs.

$x \xrightarrow{LIST} LIST(x)$ $\qquad$ $x \xrightarrow{SHIP} SHIP(x)$ $\qquad$ $x \xrightarrow{PAY} PAY(x)$



binomial function

8.2 Binomial Functions

1. Given a *base* function which is a monomial function, when we *add-on* a monomial function with the *same* exponent, the *sum* is a monomial function with the same exponent.

EXAMPLE 8.3. Given the base function $MINT$ specified by the *global input-output rule*

$$x \xrightarrow{MINT} MINT(x) = -12.82x^{+4}$$

and given the add-on function TEA specified by the *global input-output rule*

$$x \xrightarrow{TEA} TEA(x) = +49.28x^{+4}$$

then the sum function will be specified by the *global input-output rule*

$$\begin{aligned} x \xrightarrow{\quad SUM \quad} SUM(x) &= MINT(x) + TEA(x) \\ &= -12.82x^{+4} \oplus +49.28x^{+4} \\ &= [-12.82 \oplus +49.28]\, x^{+4} \\ &= +36.46x^{+4} \end{aligned}$$

2. However, when the exponent of the *add-on* function is different from the exponent of the *base* function, then the *sum* function is not a *monomial function* but what is called a **binomial function**.

EXAMPLE 8.4. Let $BASE$ be specified by the global input-ouput rule

$$x \xrightarrow{BASE} BASE(x) \; = (-3)x^{+2}$$

and let $ADD\text{-}ON$ be specified by the global input-ouput rule

$$\begin{aligned} x \xrightarrow{ADD\text{-}ON} ADD\text{-}ON(x) &= (+5)x^{0} \\ &= +5 \end{aligned}$$

then the SUM function is specified by the global input-ouput rule

$$\begin{aligned} x \xrightarrow{SUM} SUM(x) &= (-3)x^{+2} \oplus (+5)x^{0} \\ &= (-3)x^{+2} + 5 \end{aligned}$$

To see that SUM *cannot* be replaced by a single monomial function, we first evaluate all three functions at some input, for instance $+2$:

$$\begin{aligned} +2 \xrightarrow{BASE} BASE(+2) &= (-3)(+2)^{+2} \\ &= -12 \end{aligned}$$

and

$$\begin{aligned} +2 \xrightarrow{ADD\text{-}ON} ADD\text{-}ON(+2 &= (+5)(+2)^{0} \\ &= +5 \end{aligned}$$

then

$$\begin{aligned} x \xrightarrow{SUM} SUM(x) &= (-3)(+2)^{+2} \oplus (+5)(+2)^{0} \\ &= -12 \oplus +5 \\ &= -7 \end{aligned}$$

The question then is what *monomial function* could return the output -7 for the input $+2$.

Of course, we can easily find a monomial function that would return the output -7 for the input $+2$. For instance, the dilation function $x \xrightarrow{f} f(x) = -\frac{7}{2}x$ does return the output -7 for the input $+2$. But$f$ is *not* going to return the same output as SUM for other inputs, say, $+3$, $+4$, etc which it should. So, the *binomial function*

$$x \xrightarrow{SUM} SUM(x) = (-3)x^{+2} + 5$$

cannot be replaced by the single *monomial function*

$$x \xrightarrow{f} f(x) = -\frac{7}{2}x$$

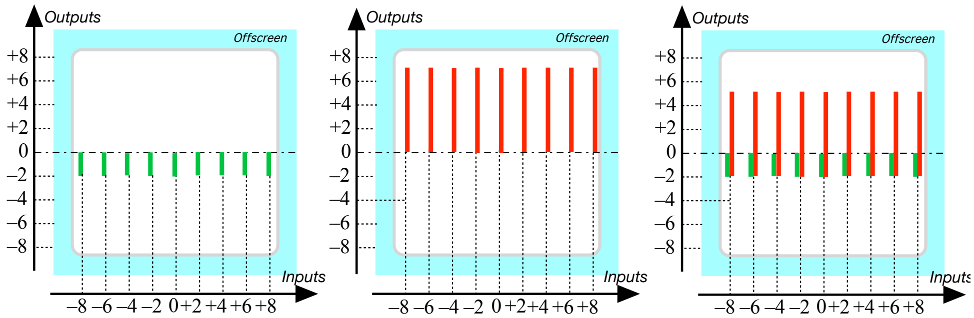
NOTE. We noted at the beginning of Chapter 5 that monomial functions were only rarely called *monomial functions* and that this was unfortunate: indeed, it would be nicer to say that a *binomial* function cannot be replaced by a single *monomial* function. (We cannot have two for the price of one.)

8.3 Graphs of Binomial Functions

1. When the exponent of the *add-on* function is the same as the exponent of the *base* function, the bar graphs show exactly why the *sum* function will have again the same exponent.

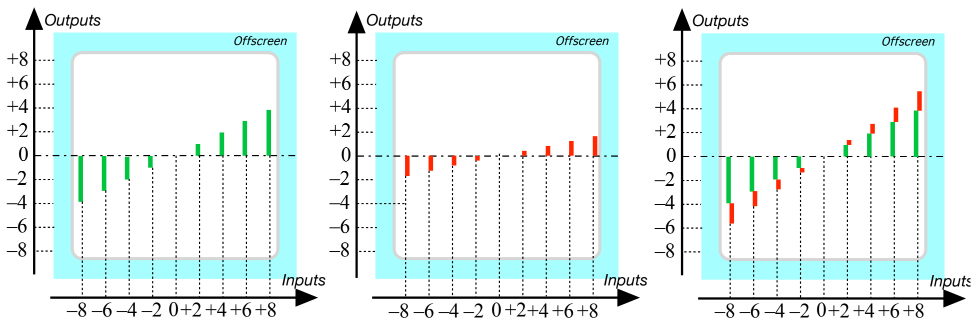
a. Given a constant base function, adding-on a constant function:

Example 8.5.



b. Given a dilation base function, adding-on a dilation function:

Example 8.6.

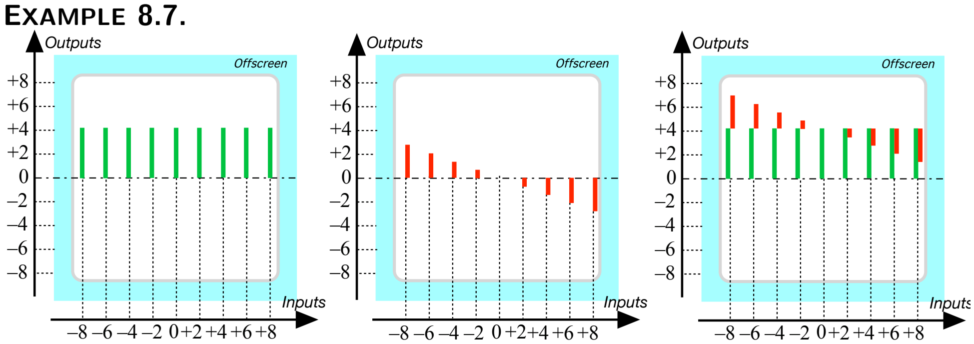


2. When the exponent of the add-on function is *not* the same as the exponent of the base function, the bar graphs show clearly why the *sum* function cannot be a monomial function.

a. Given a constant base function,

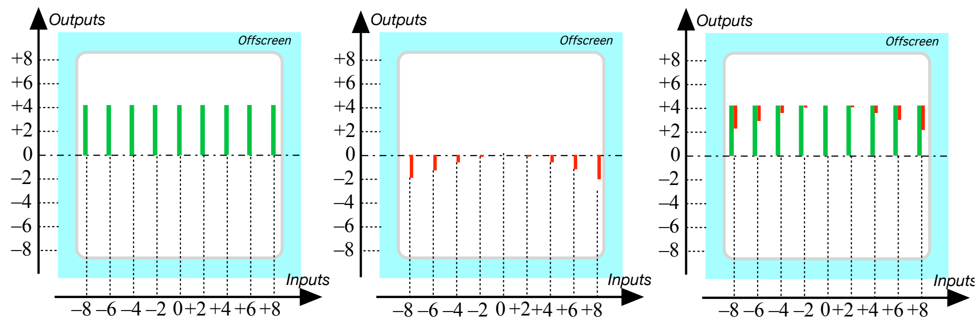
- Adding-on a dilation function:

  Example 8.7.



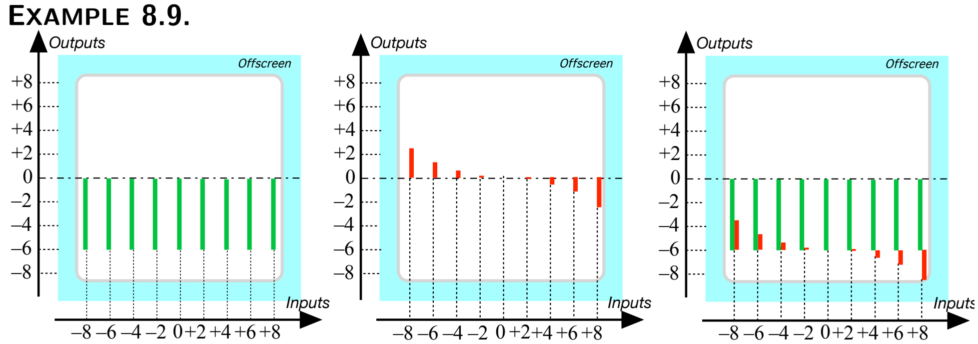
- Adding-on an even positive exponent monomial function:

  Example 8.8.



- Adding-on an odd positive exponent monomial function:

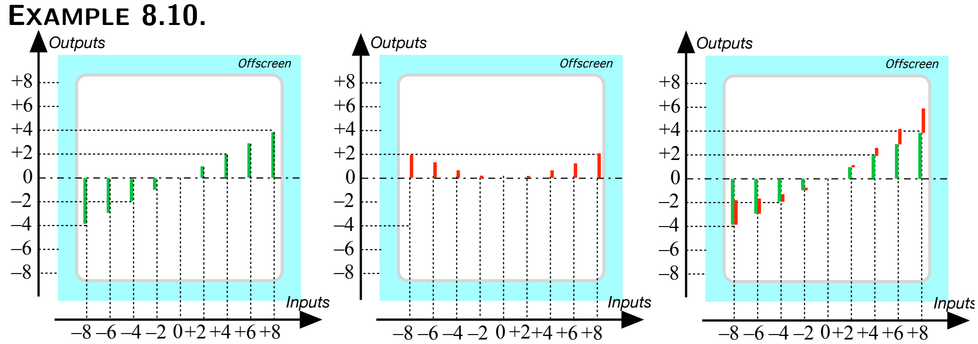
EXAMPLE 8.9.



b. Given a dilation base function,

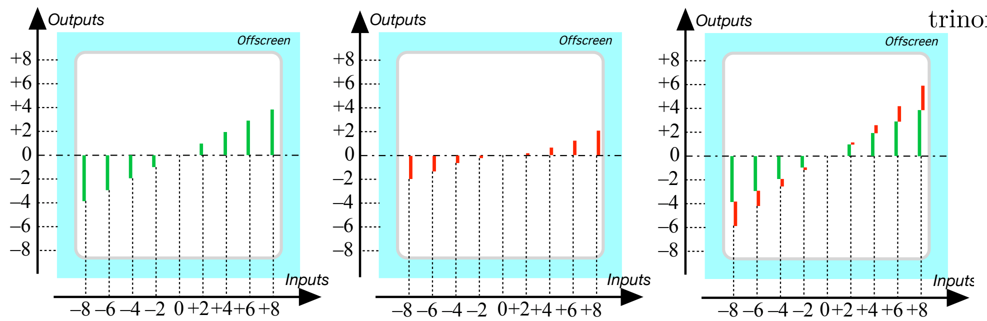
- Adding-on an even monomial function:

EXAMPLE 8.10.



- Adding-on an odd monomial function:

EXAMPLE 8.11.



trinomial function

8.4 Trinomial Functions

There is of course no reason why the base function could not itself be a binomial function. In fact, this can very well be the case and the sum function will then be called a **trinomial function**.

EXAMPLE 8.12. Let $BASE$ be specified by the global input-ouput rule

$$x \xrightarrow{BASE} BASE(x) = (-3)x^{0} \oplus (+7)x^{+1}$$

and let $ADD\text{-}ON$ be specified by the global input-ouput rule

$$x \xrightarrow{ADD\text{-}ON} ADD\text{-}ON(x) = (+5)x^{+3}$$

then the SUM function is specified by the global input-ouput rule

$$\begin{aligned} x \xrightarrow{SUM} SUM(x) &= (-3)x^{0} \oplus (+7)x^{+1} \oplus (+5)x^{+3} \\ &= -3 \; + 7x \; + 5x^{+3} \end{aligned}$$

EXAMPLE 8.13. Let $BASE$ be specified by the global input-ouput rule

$$x \xrightarrow{BASE} BASE(x) = (-3)x^{+1} \oplus (+7)x^{0}$$

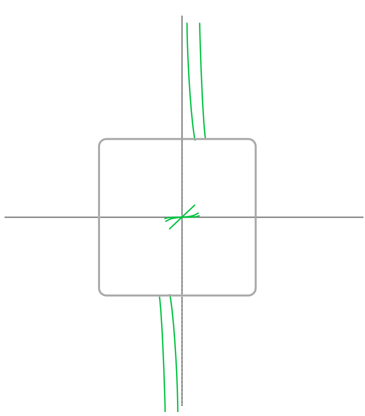
and let $ADD\text{-}ON$ be specified by the global input-ouput rule

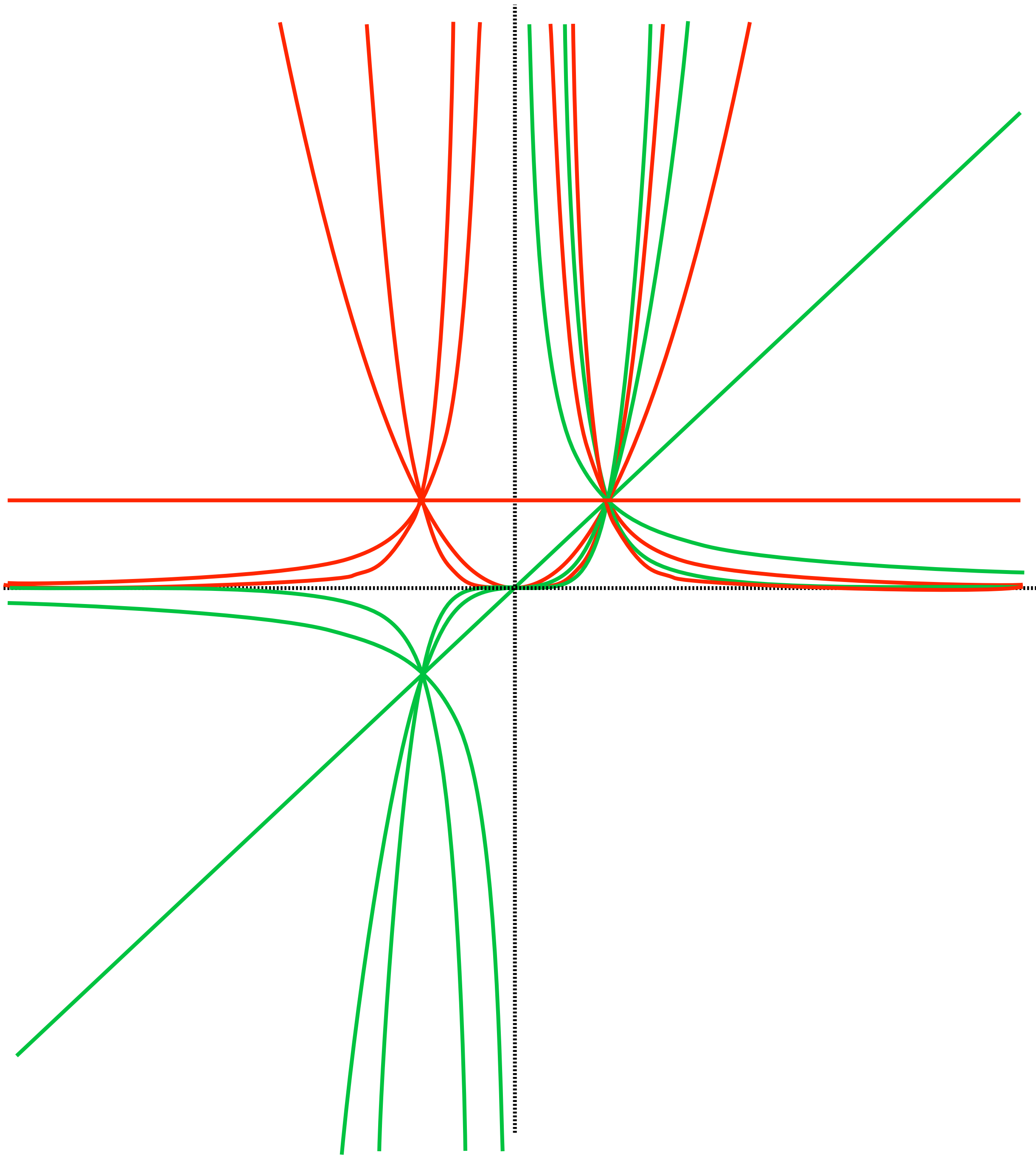
$$x \xrightarrow{ADD\text{-}ON} ADD\text{-}ON(x) = (+5)x^{-2}$$

then the SUM function is specified by the global input-ouput rule

$$\begin{aligned} x \xrightarrow{SUM} SUM(x) &= (-3)x^{+1} \oplus (+7)x^{0} \oplus (+5)x^{-2} \\ &= -3x \; + 7 \; + 5x^{-2} \end{aligned}$$

8.5 Comparing Monomial Functions





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