Chapter 3

Graphic Analysis

3.1 Local Graph Near A Given Input

Given a function $f$ and given an input, $\infty$ or a finite number $x_0$, we can observe the behavior of $f$ near the given input on the local graph of $f$ near the given input, that is the part of the global graph that corresponds...
to a neighborhood of the given input. And indeed, in our investigation of functions specified by a *global input-output rule* in later chapters, we will always begin by getting the local graph from the *global input-output rule*.

1. Here, though, since functions will be specified by a *global graph*, what we will do is use the procedure:

   5. *To get the local graph near a given input for a function specified by a global graph:*

   i. We highlight a *neighborhood* of the given input on the input ruler,
   ii. We imagine plotting the input-output pairs for *nearby inputs*,
   iii. We highlight the part of the *global graph* where the plot points would be.

2. Even though we are plotting input-output pairs, this procedure has absolutely nothing to do with the so-called procedure we criticized in Chapter 2, Section 6 because here we already have the global graph.

3. Even though the procedure is the same whether the given input near which we want to get the local graph is $\infty$ or a finite number $x_0$, we will deal with the two cases separately because:
   - When we will get to “coding” the local behavior near the given input, we will have to face the center of the neighborhood, that is the given input, and whether the given input is $\infty$ or a finite number $x_0$ will make a difference.
   - We will often have to look at one *side* of the local graph:
     - The *left side* of the *local graph* which is for inputs *left* of the given input (when facing the center of the neighborhood),
     or
     - The *right side* of the *local graph* which is for inputs *right* of the given input (when facing the center of the neighborhood).

3.2 *Local Graph Near A Finite Input* $x_0$

Let $f$ be a function specified by a global graph. The *local graph of $f$ near a finite input* $x_0$ is the part of the global graph for inputs that are near $x_0$.

1. When we need the whole local graph, following the procedure is completely straightforward.
3.2. LOCAL GRAPH NEAR $X_0$

**Example 1.** Let $f$ be the function specified by the global graph

and let the given input be $+3$.

To get the local graph of $f$ near $+3$:

i. We highlight a neighborhood of $+3$

ii. We imagine plotting input-output pairs for a few nearby inputs,

iii. We highlight the part of the global graph on which the plot points would be.

2. When we need just one side of the local graph, things are still straightforward:
Since we are facing the screen, we automatically face the given input that is the center of the neighborhood so that the left side of the local graph will be to our left and the right side of the local graph will be to our right.

**Example 2.** Let $f$ be the function specified by the global graph

and let the given input be $+3$.

To get the left side of the local graph of $f$ near $+3$:

i. We highlight the left side of a neighborhood of $+3$ — which is of course to our left.

ii. We imagine plotting input-output pairs for a few nearby inputs on the left side. (on the enlargement),

iii. We highlight the part of the global graph on which the plot points would be.
3.3 Local Graph Near Infinity

Let $f$ be a function specified by a global graph. The local graph of $f$ near $\infty$ is the part of the global graph for inputs that are near $\infty$. But, while we will still use the procedure, because the given input is $\infty$ there will be two complications:

- We will have to take into account the fact that, since we can only be facing the screen, we cannot be facing the center of the neighborhood, namely $\infty$.
- We will have to take into account the fact that, as we saw in 1.8 The Two Sides of Infinity, there are two different ways to distinguish and refer to infinitely large-in-size numbers on the two sides of $\infty$.

Correspondingly, there are two ways to draw the local graph near $\infty$. More precisely,

\[\text{---------OK SO FAR---------}\]

i. When we use the sign to refer to the side

**Example 3.** Given the function whose global graph is its local graph near $\infty$ is

\[\text{---------OK SO FAR---------}\]

ii. 1.
2. Before we can deal with the *sides* of a local graph near $\infty$, we must discuss a difference with local graphs near a *finite* input $x_0$.

- When the given input is a *finite* input $x_0$, we are automatically facing the center $x_0$ of the neighborhood since we are facing the screen:

- When the given input is $\infty$, since we are facing the screen we are not facing the center of the neighborhood. So, to face $\infty$, we have to see the input ruler as part of a *Magellan circle* and imagine ourselves "down under" so that $+\infty$ is *to our left* and $-\infty$ is *to our right* (as opposed to when we are facing the screen):

3. Thus, while facing $\infty$:
- The *local graph* near $+\infty$ is the *local graph* for inputs *left* of $\infty$,
- The *local graph* near $-\infty$ is the *local graph* for inputs *right* of $\infty$.

**Example 4.** Local graph *left* of $\infty$:

**Local graph right of $\infty$**:

### 3.4 Local Code

Given a function and given an input, which can be $\infty$ or a *finite input* $x_0$, we will want to describe the features of the *local graph* near the given input and so we will need a **local code** in which to write these local features.

1. Since there is no reason for the local behavior to be the same on both sides of the given input, the local code will have to take care *separately* of the features *left* of the given input and of the features *right* of the given input.
3.5. PLACE OF A LOCAL GRAPH

The first two local features that we will be dealing with are the size and the sign of the outputs for inputs near the given input:

1. The height-size of a given input on a given side is the size of the outputs for inputs that are near the given input on that side.

We will code the height-size
- with \( \infty \) to say that the height-size is infinite,
- with \( \flat \) to say that the height-size is finite but not small,
- with 0 to say that the height-size is small.

Example 5. Given the function \( JANE \)
whose local graph near $+5$ is

- the height size on the left side of $+5$ is infinite
- the height size on the right side of $+5$ is small

which we code as follows:

\[ \text{Height Size } JANE \text{ near } +5 = (\infty, 0) \]

2. The **height-sign** of a given input on a given side is the sign of the outputs for inputs that are near the given input on the given side.

We will code the height-sign

- with $+$ to say that the local graph is above 0,
- with $-$ to say that the local graph is below 0.

**Example 6.** Given the function $ZOE$

whose local graph near $\infty$ is

- the height sign on the left side of $\infty$ is $-$
- the height sign on the right side of $\infty$ is $+$

which we code as follows:

\[ \text{Height Sign } ZOE \text{ near } \infty = (-, +) \]

3. The **Height Sign-Size** of a local graph is the height sign together with the height size.

When coding the Height Sign-Size, though, we will have to keep in mind an unfortunate linguistic “peculiarity”, namely that:

- When the code for the size is $\infty$, the code for the sign is written before the code for the size just as if it were a signed numbers: $+\infty$, $-\infty$.
- But when the code for the size is 0 or $\flat$, the code for the sign is written after the code for the size just as if it were an exponent: $0^+$, $0^-$, $\flat^+$, $\flat^-$. 

**Example 7.** Given the function $ZACH$
whose local graph near $\infty$ is

- the height sign-size on the left side of $\infty$ is \emph{large} noteb
- the height sign-size on the right side of $\infty$ is \emph{small} noteb
which we code as follows:

$\text{Height Sign-Size } \text{ZACH } \text{near } \infty = (\infty, 0^-)$

4. Together, the \emph{Height Sign} and the \emph{Height Size} give us the \textbf{place} of the local graph.

\textbf{Example 8.} If the Height Sign-Size of a function $f$ near $+\infty$ is $-\infty$ then the local graph of $f$ near $+\infty$ is in the following place:

\textbf{Example 9.} If the Height Sign-Size of a function $f$ near $-\infty$ is $0^+$ then the local graph of $f$ near $-\infty$ is in the following place:

\textbf{Example 10.} If the Height Sign-Size of a function $f$ near $-5$ is $(0^-, +\infty)$ then the local graph of $f$ near $-5$ is in the following place:

3.6 \textit{$\infty$}-Height Inputs and 0-Height Inputs

Related to the \textit{height-size} of the local graph, there are two kinds of \textbf{notable inputs}:

- A \textit{finite} input $x_0$ is an \textbf{$\infty$-height input} if inputs that are near $x_0$ have
infinite outputs. We will use $x_{\infty}$-height to refer to $\infty$-height finite input.

- A finite input $x_0$ is an 0-height input if inputs that are near $x_0$ have small outputs. We will use $x_0$-height to refer to 0-height finite input.

Both $\infty$-height inputs and 0-height inputs can be:

- **even** if the height-sign remains the same on both side
- **odd** if the height-sign changes from one side to the other.

**Example 11.** The following are $\infty$-height finite inputs

**Example 12.** The following are 0-height finite inputs

### 3.7 Shape of a Local Graph

We now introduce two local features of a local graph that refer to the **shape** of the local graph. Here, though, we will only be concerned with the qualitative aspect and so we will only deal with the **sign** of the features and not with their **size**.

1. The **slope sign** of the local graph near $x_0$ says whether the local graph near $x_0$ is
   - sloping up, that is the local graph looks more or less like / in which case we will also say that the slope is **positive**
3.7. SHAPE OF A LOCAL GRAPH

- **sloping down**, that is the local graph looks more or less like \ in which case we will also say that the **slope is negative**

Even though + and − are the symbols that are used traditionally, here, for the sake of “transparency”, we will **code the slope sign**
- with / to say that the local graph is **going up (positive slope)**,
- with \ to say that the local graph is **going down (negative slope)**.

**EXAMPLE 13.** Following are five local graphs together with their slope-sign:

- **Sloping Up**
- **Sloping Up**
- **Sloping Up**
- **Sloping Down**
- **Sloping Up**

Slope-sign = (✓✓)
Slope-sign = (✓✓)
Slope-sign = (✓✓)
Slope-sign = (✓✓)
Slope-sign = (✓✓)

**EXAMPLE 14.** What is the slope-sign near +2 of the function \( f \) specified by the curve,

We read the slope-sign from the local graph (blown up here for convenience)

Slope-Sign \( ABEL \) near +2 = (✓✓)

2. The **concavity sign** of the local graph near \( x_0 \) says whether the local graph near \( x_0 \) is
- **bending up**, that is the local graph looks like part of a **cup** like \( \cup \)
- **bending down**, that is the local graph looks like part of a **cap** like \( \cap \)
Even though + and − are the symbols that are used traditionally, here, for the sake of “transparency”, we will code the concavity sign
• with \( \cup \) to say that the local graph is bending up (positive concavity),
• with \( \cap \) to say that the local graph is bending down (negative concavity).

**Example 15.** Following are five local graphs together with their concavity-sign:

\[
\text{Concave Down Up} \quad \text{Concave Down Down} \quad \text{Concave Up Down} \quad \text{Concave Down Up} \quad \text{Concave Down Up}
\]

Conc-sign = (\( \cap, \cup \))  Conc-sign = (\( \cap, \cap \))  Conc-sign = (\( \cup, \cap \))  Conc-sign = (\( \cap, \cup \))  Conc-sign = (\( \cup, \cup \))

**Example 16.** What is the concavity-sign near \(-1\) of the function \( f \) specified by the curve

We read the concavity-sign from the local graph (blown up here for convenience)

Concavity-Sign \( f \) near \(-1\) = (\( \cup, \cap \))
3.8 Feature-Sign Change Inputs

Given a feature, a sign-change input for that feature is an input for which the sign of the feature is different on the two sides of that input.¹

1. Given a function \( f \) and an input \( x_0 \), we will say that \( x_0 \) is a height-sign change input when height-sign is different on the two sides of \( x_0 \).

Example 17. Given the function \( JANE \) and given the input \(-4\),

\[ \text{Height-Sign near } -4 = ( -, + ) \]

\(-4\) is a height-sign change input.

Example 18. Given the function \( KANE \) and given the input \(+1\),

\[ \text{Height-Sign near } +1 = ( +, - ) \]

\(+1\) is a height-sign change input.

2. Given a function \( f \) and an input \( x_0 \), we will say that \( x_0 \) is a slope-sign change input when slope-sign is different on the two sides of \( x_0 \).

Example 19. Given the function \( MARY \) and given the input \(+5\)

¹Educologists will surely cringe at this terminology even though, if nothing else, it has the double merit of being systematic and self-explanatory.
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concavity-sign change input

+5 is a slope-sign change input.

**Example 20.** Given the function $LARS$ and given the input +1,

\[ \text{Output Ruler} \]
\[ \text{Input Ruler} \]
\[ \text{Screen} \]
\[ +0 +1 +2 +3 +4 +5 –1 \]
\[ +1 ( ) \]
\[ \text{Slope-Sign near +1} = ( ), \]
\[ +5 \text{ is not a slope-sign change input.} \]

3. Given a function $f$ and an input $x_0$, we will say that $x_0$ is a **concavity-sign change input** when concavity-sign is different on the two sides of $x_0$.

**Example 21.** Given the function $NATE$ and given the input +5

\[ \text{Output Ruler} \]
\[ \text{Input Ruler} \]
\[ \text{Screen} \]
\[ +6 +7 +4 +5 +5 \]
\[ ( ) \]
\[ \text{Concavity Sign near +5} = ( ), \]
\[ +5 \text{ is not a concavity-sign change input.} \]

**Example 22.** Given the function $PETE$ whose local graph near +1 is

\[ \text{Output Ruler} \]
\[ \text{Input Ruler} \]
\[ \text{Screen} \]
\[ +0 +1 +2 +3 +4 +5 –1 \]
\[ +1 ( ) \]
\[ \text{Concavity Sign near +1} = ( ), \]
\[ +1 \text{ is a concavity-sign change input.} \]
3.9 0-Slope and 0-Concavity Inputs

Related to the slope and the concavity of the local graph, the following are the third kind of notable inputs we will be looking for:

1. A finite input $x_0$ is an 0-slope input if inputs that are near $x_0$ have small slope. We will use $x_{0\text{-slope}}$ to refer to 0-slope inputs.

   **Example 23.** The following are two examples of 0-slope finite inputs

2. A finite input $x_0$ is an 0-concavity input if inputs that are near $x_0$ have small concavity. We will use $x_{0\text{-concavity}}$ to refer to 0-concavity inputs.

   **Example 24.** The following are two examples of $\infty$-concavity finite inputs

3.10 Extreme-Height Inputs

In many applications to the real world, one needs to compare the height of a given finite input to the height of nearby inputs. An extreme-height input is a finite input whose output is either absolutely larger than the height of all nearby inputs or absolutely smaller than the height of all nearby inputs.

- When the height of the extreme-height input $x_0$ is absolutely larger than the height of all nearby inputs, $x_0$ is called a maximum-height input.

   We will use $x_{\text{maximum-height}}$ to refer to maximum-height inputs.

   From the graphic viewpoint, the local graph near a maximum-height input is entirely below the output-level line for the maximum-height input.
minimum-height input $x_{\text{minimum-height}}$

**Example 25.**

- When the height of the extreme-height input $x_0$ is absolutely smaller than the height of all nearby inputs, $x_0$ is called a minimum-height input. We will use $x_{\text{minimum-height}}$ to refer to minimum-height inputs.

From the graphic viewpoint, the local graph near a minimum-height input is entirely above the output-level line for the minimum-height input.

**Example 26.**
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