REVIEWS AND EXAMS
For each one of the three parts of the Text, the following are available:

- A Review Homework with open spaces for the students to investigate each one of twenty-five questions in text order.
- A Review Discussion in which the twenty-five questions in the corresponding Review Homework are being discussed at some length.
- A Review Pretest in which the twenty-five questions in the corresponding Review Homework appear in text or random order with multiple-choice answers.
- An Exam in which random multiple-choice avatars of the twenty-five questions in the corresponding Review Homework appear in random order.

Following are just the three Review Discussions.
1. **Question:** Given the number-phrases \(-17.84\) **Dollars** and \(+6.37\) **Dollars** in that order, which *lenient* size-comparison sentences is *true*?  

**Discussion:** A *size*-comparison sentence involves only the *size* and not the sign of the signed number-phrases being compared.  

We can look at the question from two points of view:  

- **In a corresponding real-world situation,** we have:  
  - Jack owes seventeen dollars and eighty-four cents,  
  - Jill is owed six dollars and thirty-seven cents  
  Since we are *size*-comparing, what matters is only how much is being owed and Jack owes no less than what Jill is owed.  
- **In the paper representation,** we compare only the *sizes* of the signed numbers.  

Either way, we end up writing the *size*-comparison sentence

\[-17.84 \text{ Dollars is-no-smaller-in-size-than} + 6.37 \text{ Dollars}\]

2. **Question:** Given the number-phrases \(-27.32\) **Dollars** and \(-742.33\) **Dollars** in that order, which *strict* algebra-comparison sentence is *true*?  

**Discussion:** An *algebra*-comparison sentence involves both the *size* and the *sign* of the signed number-phrases being compared.  

We can look at the question from two points of view:  

- **In a corresponding real-world situation,** we have:  
  - Jack owes twenty-seven dollars and eighty-four cents  
  - Jill owes seven hundred, forty-two dollars and thirty-three cents  
  Since we are *algebra*-comparing, what matters is that, since Jack *owes less* than Jill, Jack is *better off* than Jill.  
- **In the paper representation,** any two *negative* number-phrase *algebra-compare* the way *opposite* to the way they *size-compare*.  

Either way, we end up writing the *algebra*-comparison sentence

\[-27.32 \text{ Dollars} > -742.33 \text{ Dollars}\]

3. **Question:** The single action that gives the same result as a twenty-three dollars and fifty-two cents deposit followed by a sixty-eight dollars
and seventy-six cents withdrawal is represented by what signed number-phrase?

**Discussion:** Depositing money and withdrawing money are real-world actions that we represent on paper by signed number-phrases. Following a first action by a second action is represented on paper by the addition of the second number-phrases to the first number-phrase.

We can look at the question from two points of view:

- In a corresponding real-world situation,
  - Jack deposits twenty-three dollars and thirty-two cents
  - Jill withdraws sixty eight dollars and thirty-three cents
Since Jill’s action is more in size than Jack’s action, she is in fact withdrawing the twenty-three dollars and thirty-two cents that Jack had deposited and another forty-five dollars and twenty-four cents.
- In the paper representation, we write the specifying number-phrase 

  \[ +23.52 \text{ Dollars} \oplus -68.76 \text{ Dollars} \]

  and then THEOREM 1 says that since the number-phrases are opposite in sign, 
  - We get the sign of the result by taking the sign of the signed number-phrase whose size is larger,
  - We get the size of the result by subtracting the smaller size from the larger size.

Either way, we end up writing the signed number-phrase 

\[ -45.24 \text{ Dollars} \]

4. **Question:** You thought your balance was one hundred seventy-two dollars and fifty-seven cents in the black but you just found out that a twelve dollars and fifty-six cents check you had deposited bounced. What is the signed number-phrase that represents your new balance?

**Discussion:** Removing a deposit or removing a withdrawal is a real-world action that is represented on paper by a subtraction.

We can look at the question from two points of view:

- In a corresponding real-world situation,
  - You thought the balance was one hundred seventy-two dollars and fifty-seven cents in the black
  - but this balance included a twelve dollars and fifty-six cents check
Since the check bounced, the balance is actually twelve dollars and fifty-six cents less than you thought, that is one hundred sixty dollars and one cent in the black.
• In the paper representation, we write the specifying-phrase
  \[ +172.57 \text{ Dollars} \bigoplus +12.56 \text{ Dollars} \]
  which we identify by adding the opposite of the second number-phrase
to the first number-phrase
  \[ +172.57 \text{ Dollars} \bigoplus -12.56 \text{ Dollars} \]
Either way, we end up writing the signed number-phrase

  \[ +160.01 \text{ Dollars} \]

5. Question: You thought your balance was one hundred seventy-two
dollars and fifty-seven cents in the red but you just found out that an
unjustified twelve dollars and fifty-six cents charge has been removed.
What is the signed number-phrase that represents your new balance?

Discussion: Removing a deposit or removing a withdrawal is a real-
world action that is represented on paper by a subtraction.

We can look at the question from two points of view:
• In a corresponding real-world situation,
  – You thought the balance was one hundred seventy-two dollars
    and fifty-seven cents in the red
  – but this balance included a twelve dollars and fifty-six cents charge
Since the charge was removed, the balance is actually twelve dollars
and fifty-six cents more than you thought, that is one hundred sixty
dollars and one cent in the red.
• In the paper representation, we write the specifying-phrase
  \[ -172.57 \text{ Dollars} \bigoplus -12.56 \text{ Dollars} \]
  which we identify by adding the opposite of the second number-phrase
to the first number-phrase
  \[ -172.57 \text{ Dollars} \bigoplus +12.56 \text{ Dollars} \]
Either way, we end up writing the signed number-phrase

  \[ -160.01 \text{ Dollars} \]

6. Question: On Monday your balance was three hundred thirty-two
dollars and seventy one cents in the red and on Thursday your balance
was seventy-four dollars and forty-six cents in the red. What is the signed
number-phrase that represents the change in your balance from Monday
to Thursday?

Discussion: The change is the single action on the initial state that
results in the final state.
We can look at the question from two points of view:

- In a corresponding *real-world situation*,
  - on Monday your balance was three hundred thirty-two dollars and seventy-one cents in the red
  - on Thursday your balance was seventy-four dollars and forty-six dollars in the red

Since, while still in the red, the balance has gone down in size, from three hundred thirty-two dollars and seventy-one cents to seventy-four dollars and forty-six dollars, this means that the action must have been a gain of two hundred fifty-eight dollars and twenty-five cents.

- In the *paper representation*, THEOREM 2 says that the change from an initial state to a final state is equal to the final state minus the initial state. So we write the specifying-phrase
  \[ -74.46 \text{ Dollars} \oplus -332.71 \text{ Dollars} \]
  that is
  \[ -74.46 \text{ Dollars} \oplus +332.71 \text{ Dollars} \]

Either way, we end up writing

\[ -258.25 \text{ Dollars} \]

7. **Question:** On Tuesday your balance was six hundred three dollars and twenty-eight cents in the red and on Friday your balance was fifty-six dollars and three cents in the black. What is the signed number-phrase that represents the change in your balance from Tuesday to Friday?

**Discussion:** The change is the single action on the initial state that results in the final state.

We can look at the question from two points of view:

- In a corresponding *real-world situation*,
  - on Tuesday your balance was six hundred three dollars and twenty-eight cents in the red
  - on Friday your balance was fifty-six dollars and three cents in the black

Since your balance has gone from the red to the black, from six hundred three dollars and twenty-eight cents in the red to fifty-six dollars and three cents in the black, this means that the action must have been a gain of six hundred three dollars and twenty-eight cents plus fifty-six dollars and three cents, that is a gain of six hundred fifty-nine dollars and thirty-one cents.
• In the paper representation, THEOREM 2 says that the change from an initial state to a final state is equal to the final state minus the initial state. So we write the specifying-phrase 

\[ +56.03 \text{ Dollars} \oplus -603.28 \text{ Dollars} \]

that is

\[ +56.03 \text{ Dollars} \oplus +603.28 \text{ Dollars} \]

Either way, we end up writing

\[ +659.31 \text{ Dollars} \]

8. Question: Your balance was seventy-six dollars and thirty-eight cents in the red and you made an eight hundred seventy-six dollars and eleven cents withdrawal. What is the signed number-phrase that represents your new balance?

Discussion: The final state is the result of the action on the initial state.

We can look at the question from two points of view:

• In a corresponding real-world situation,
  – The initial state of your account was seventy-six dollars and thirty-eight cents in the red
  – The action on this initial state was an eight hundred and seventy-six dollars and eleven cents withdrawal.

Since your are withdrawing money from an account that was already in the red, the eight hundred and seventy-six dollars and eleven cents add to the seventy-six dollars and thirty-eight cents to give a final balance of nine hundred fifty-two dollars and forty-nine cents in the black.

• In the paper representation, we write the signed specifying-phrase

\[ -76.38 \text{ Dollars} \oplus -876.11 \text{ Dollars} \]

and we identify it.

Either way, we end up writing

\[ +952.49 \text{ Dollars} \]

9. Question: Your balance was seventy-six dollars and thirty-eight cents in the black and you made an eight hundred seventy-six dollars and eleven cents withdrawal. What is the signed number-phrase that represents your new balance?
Discussion: The final state is the result of the action on the initial state.

We can look at the question from two points of view:

- In a corresponding real-world situation,
  - The initial state of your account was seventy-six dollars and thirty-eight cents in the black
  - The action on this initial state was an eight hundred and seventy-six dollars and eleven cents withdrawal.

Since your are withdrawing more money than was in the account, the eight hundred and seventy-six dollars and eleven cents break down to the seventy-six dollars and thirty-eight cents that were in the account and the remainder that gives a final balance of seven hundred ninety-nine dollars and seventy-three cents in the red.

- In the paper representation, we write the signed specifying-phrase

\[ +76.38 \text{ Dollars} \oplus -876.11 \text{ Dollars} \]

and we identify it.

Either way, we end up writing

\[-799.73 \text{ Dollars} \]

10. Question: Your balance was seventy-six dollars and thirty-eight cents in the red and you made an eight hundred seventy-six dollars and eleven cents deposit. What is the signed number-phrase that represents your new balance?

Discussion: The final state is the result of the action on the initial state.

We can look at the question from two points of view:

- In a corresponding real-world situation,
  - The initial state of your account was seventy-six dollars and thirty-eight cents in the red
  - The action on this initial state was an eight hundred and seventy-six dollars and eleven cents deposit.

Since your are depositing money from an account that was already in the red, the eight hundred and seventy-six dollars and eleven cents first go to the seventy-six dollars and thirty-eight cents in the red to give a final balance of seven hundred ninety-nine dollars and seventy-three cents in the black.

- In the paper representation, we write the signed specifying-phrase
−76.38 Doll$\$s \oplus +876.11 Doll$\$s$

and we identify it.

Either way, we end up writing

+799.73 Doll$\$s$

**11. Question:** Your balance was seventy-six dollars and thirty-eight cents in the black and you made an eight hundred seventy-six dollars and eleven cents deposit. What is the signed number-phrase that represents your new balance?

**Discussion:** The final state is the result of the action on the initial state.

We can look at the question from two points of view:

- In a corresponding real-world situation,
  - The initial state of your account was seventy-six dollars and thirty-eight cents in the black
  - The action on this initial state was an eight hundred and seventy-six dollars and eleven cents deposit.

Since you are depositing money on an account that was already in the black, the eight hundred and seventy-six dollars and eleven cents add to the seventy-six dollars and thirty-eight cents to give a final balance of *nine hundred fifty-two dollars and forty-nine cents in the black*.

- In the paper representation, we write the signed specifying-phrase

  +76.38 Doll$\$s \oplus +876.11 Doll$\$s$

  and we identify it.

Either way, we end up writing

+952.49 Doll$\$s$

**12. Question:** What is the distance between −332.71 Doll$\$s and −74.46 Doll$\$s

**Discussion:** The distance between two signed number-phrases is the size of the change from either one to the other.

We obtain the change from −332.71 Doll$\$s to −74.46 Doll$\$s by subtracting the first signed number-phrase from the second number-phrase:

−74.46 Doll$\$s \ominus −332.71 Doll$\$s
that is

$$-74.46 \text{ Dollars} \oplus +332.71 \text{ Dollars}$$

which gives the change

$$+258.25 \text{ Dollars}$$

whose size is

$$258.25 \text{ Dollars}$$

If we compute the change from $$-74.46 \text{ Dollars} to -332.71 \text{ Dollars}$$, we get the opposite change:

$$-258.25 \text{ Dollars}$$

whose size, though, is still the same:

$$258.25 \text{ Dollars}$$

13. Question: What is the distance between $$-332.71 \text{ Dollars} and +74.46 \text{ Dollars}$$?

Discussion: The distance between two signed number-phrases is the size of the change from either one to the other. We obtain the change from $$-332.71 \text{ Dollars to } +74.46 \text{ Dollars}$$ by subtracting the first signed number-phrase from the second number-phrase:

$$+74.46 \text{ Dollars} \ominus -332.71 \text{ Dollars}$$

that is

$$+74.46 \text{ Dollars} \oplus +332.71 \text{ Dollars}$$

which gives the change

$$+407.17 \text{ Dollars}$$

whose size is still the same.
If we compute the change from +74.46 Dollars to −332.71 Dollars, we get the opposite change:

\[-407.17 \text{ Dollars}\]

whose size, though, is the same:

\[407.17 \text{ Dollars}\]

14. **Question:** Plot the number phrase(s) that is/are at a 3 Dollars distance from −1 Dollars.

**Discussion:** The distance between two signed number-phrases is the size of the change from either one to the other.

We can proceed in either one of two ways:

- From the algebraic viewpoint, we say that, since the distance of the final signed-number-phrase from the initial signed number-phrase −1 Dollars is required to be 3 Dollars, then the change can be either
  * positive, that is +3 Dollars, and the final signed number-phrase is:

\[
\text{Initial } \oplus \text{ positive change } = -1 \text{ Dollars } \oplus +3 \text{ Dollars} \\
= +2 \text{ Dollars}
\]

or

* negative, that is −3 Dollars, and the final signed number-phrase is:

\[
\text{Initial } \oplus \text{ negative change } = -1 \text{ Dollars } \oplus -3 \text{ Dollars} \\
= -4 \text{ Dollars}
\]

both of which we then plot.

- From the graphic viewpoint, we count from the given initial number-phrase −1 Dollars a distance of 3 Dollars in both directions.

Either way, we end up with
15. Question: Plot the number phrase(s) that is/are at a 2 Dollars distance from +3 Dollars.

Discussion: The distance between two signed number-phrases is the size of the change from either one to the other.

We can proceed in either one of two ways:

- From the algebraic viewpoint, we say that, since the distance of the final signed-number-phrase from the initial signed number-phrase +3 Dollars is required to be 2 Dollars, then the change can be either
  * positive, that is +2 Dollars, and the final signed number-phrase is:

\[
\text{Initial } \oplus \text{ positive change } = +3 \text{ Dollars } \oplus +2 \text{ Dollars } = +5 \text{ Dollars}
\]

or

* negative, that is −2 Dollars, and the final signed number-phrase is:

\[
\text{Initial } \oplus \text{ negative change } = +3 \text{ Dollars } \oplus -2 \text{ Dollars } = +1 \text{ Dollars}
\]

both of which we then plot.

- From the graphic viewpoint, we count from the given initial number-phrase +3 Dollars a distance of 2 Dollars in both directions.

Either way, we end up with

16. Question: Identify the specifying-phrase \((-116.72) \oplus (-54.07)\)

Discussion: We use, from Chapter 5, the following

**THEOREM 1.** To add signed-numerators:

- When the two signed number-phrases have the same sign,
  - We get the sign of the result by taking the common sign
  - We get the size of the result by adding the two sizes.

- When the two signed number-phrase have opposite signs, we must first compare the sizes of the two signed number-phrases and then
  - We get the sign of the result by taking the sign of the signed number-phrase whose size is larger,
We get the size of the result by subtracting the smaller size from the larger size.

Since, here, the two signed numerators have the same sign —

- we get the sign of the result by taking the common sign —
- we get the size of the result by adding the two sizes: 116.72 + 54.07

Altogether, we have identified the specifying-phrase \((-116.72) \oplus (-54.07)\) as

\[-170.79\]

17. **Question:** Identify the specifying-phrase \(-395.82 \oplus +47.93\)

**Discussion:** We use, from Chapter 5, the following

**THEOREM 1.** To add signed-numerators:

1. When the two signed number-phrases have the same sign,
   - We get the sign of the result by taking the common sign
   - We get the size of the result by adding the two sizes.

2. When the two signed number-phrase have opposite signs, we must first compare the sizes of the two signed number-phrases and then
   - We get the sign of the result by taking the sign of the signed number-phrase whose size is larger,
   - We get the size of the result by subtracting the smaller size from the larger size.

Since, here, the two signed numerators have opposite signs, we must compare the sizes of the two numerators.

- we get the sign of the result by taking the sign of \(-395.82\) since it is larger in size than \(+47.93\).
- we get the size of the result by subtracting the smaller size from the larger size: 395.82 – 47.93

Altogether, we have identified the specifying-phrase \(-395.82 \oplus +47.93\) as

\[-347.89\]

18. **Question:** Identify the specifying-phrase \(+496.81 \ominus -52.59\)

**Discussion:** To subtract a signed number-phrase means to add the opposite of this signed number-phrase.

So,

\[+496.81 \ominus -52.59 = +496.81 \oplus +52.59\]
19. Question: Identify the specifying-phrase \( +728.22 \ominus +76.29 \)

Discussion: To **subtract** a signed number-phrase means to **add the opposite** of this signed number-phrase.

So,

\[
+728.22 \ominus +76.29 = +728.22 \oplus -76.29 \\
= +651.93
\]

20. Question: Identify \( 2 - 1 + 4 - 1 - 3 + 5 - 3 - 2 + 1 + 6 - 1 + 5 + 2 \)

Discussion:

i. The symbol \( \ominus \) goes without saying,

ii. The symbols \( + \) and \( - \) are the signs of the signed numerators,

iii. If the first numerator has no sign, the sign \( + \) goes without saying.

\[
2 - 1 + 4 - 1 - 3 + 5 - 3 - 2 + 1 + 6 - 1 + 5 + 2
\]

\[
\begin{array}{c}
+2 \oplus -1 \\
+1 \oplus +4 \\
-3 \oplus -1 \\
-4 \oplus -3 \\
-7 \oplus +5 \\
-2 \oplus -3 \\
-5 \oplus -2 \\
-7 \oplus +1 \\
-6 \oplus +6 \\
0 \oplus -1 \\
-1 \oplus +5 \\
+4 \oplus +2 \\
+6
\end{array}
\]
21. Question: Identify $-1 - 1 + 2 + 2 - 3 - 3 + 4 + 4 - 5 - 5 + 6 + 6$

Discussion:

i. The symbol $\oplus$ goes without saying,

ii. The symbols $+$ and $-$ are the signs of the signed numerators,

iii. If the first numerator has no sign, the sign $+$ goes without saying.

\[ -1 - 1 + 2 + 2 - 3 - 3 + 4 + 4 - 5 - 5 + 6 + 6 \]

\[
\begin{array}{c}
-1 \oplus -1 \\
-2 \oplus +2 \\
0 \oplus +2 \\
+2 \oplus -3 \\
-1 \oplus -3 \\
-4 \oplus +4 \\
0 \oplus +4 \\
+4 \oplus -5 \\
-1 \oplus -5 \\
-6 \oplus +6 \\
0 \oplus +6 \\
+6 \\
\end{array}
\]

22. Question: What should you add to $-3$ Dollars in order to get $+7$ Dollars?

Discussion: A simple way to do this is first to get $0$ Dollars:

i. Starting with $-3$ Dollars and in order to get $0$ Dollars we must add the opposite of $-3$ Dollars, that is $+3$ Dollars.

ii. Starting now with $0$ Dollars and in order to get $+7$ Dollars we must add $+7$ Dollars.

So, altogether, starting with $-3$ Dollars and in order to get $+7$ Dollars we must add:

\[ +3 \text{ Dollars} \oplus +7 \text{ Dollars} = +10 \text{ Dollars} \]
We check that

\[-3 \text{ Dollars} \oplus +10 \text{ Dollars} = +7 \text{ Dollars}\]

### 23. Question:
What should you *subtract* from $-3 \text{ Dollars}$ in order to get $+7 \text{ Dollars}$?

**Discussion:** To ask:

What should we subtract from $-3 \text{ Dollars}$ in order to get $+7 \text{ Dollars}$

is the same as to ask:

What should we add to $+7 \text{ Dollars}$ in order to get $-3 \text{ Dollars}$

So then,

i. Starting with $+7 \text{ Dollars}$ and in order to get $0 \text{ Dollars}$ we must add the opposite of $+7 \text{ Dollars}$, that is $-7 \text{ Dollars}$.

ii. Starting now with $0 \text{ Dollars}$ and in order to get $-3 \text{ Dollars}$ we must add $-3 \text{ Dollars}$.

So, altogether, starting with $+7 \text{ Dollars}$ and in order to get $-3 \text{ Dollars}$ we must add:

\[-7 \text{ Dollars} \oplus -3 \text{ Dollars} = -10 \text{ Dollars}\]

We check that

\[-3 \text{ Dollars} \oplus -10 \text{ Dollars} = -3 \text{ Dollars} \oplus +10 \text{ Dollars} \]

\[= +7 \text{ Dollars}\]

### 24. Question:
Identify the specifying-phrase $[+4 \text{ Apples}] \times \left[ -2 \frac{\text{Dimes}}{\text{Apple}} \right]$.

**Discussion:** We can look at the question from two points of view:

- In a corresponding *real-world situation,*
  - We have four apples appearing into the warehouse
  - Each of these apples are bad apples and will cost two dimes per apple to get rid of.

  Altogether then, this is going to cost eight dimes to the business.

- In the *paper representation,* we *co-multiply:*
  - we multiply the *denominators* (with cancellation):

\[
\text{Apples} \times \frac{\text{Dimes}}{\text{Apple}} = \text{Dimes}
\]
ii. we multiply the sizes of the numerators
\[4 \times 2 = 8\]

iii. we multiply the signs of the numerators
\((+ \otimes (-))\) gives \((-)\)

Either way, we have identified the specifying-phrase \([+4 \text{ Apples}] \times \left[-2 \frac{\text{Dimes}}{\text{Apple}}\right]\) as

\[-8 \text{ Dimes}\]

25. **Question:** Identify the specifying-phrase \([-5 \text{ Carrots}] \times \left[-7 \frac{\text{Cents}}{\text{Carrot}}\right]\)

**Discussion:** We can look at the question from two points of view:

- In a corresponding *real-world situation*,
  - We have five carrots disappearing from the warehouse
  - These carrots were bad carrots and would have cost seven cents per carrot to get rid of.
  Altogether then, this is going to be a gain of thirty-five cents for the business.
- In the *paper representation*, we co-multiply:
  i. we multiply the denominators (with cancellation):
     \[
     \frac{\text{Carrots}}{} \times \frac{\text{Cents}}{\text{Carrot}} = \text{Cents}
     \]
  ii. we multiply the sizes of the numerators
     \[5 \times 7 = 35\]
  iii. we multiply the signs of the numerators
     \((- \otimes -)\) gives \((+)\)

Either way, we have identified the specifying-phrase \([-5 \text{ Carrots}] \times \left[-7 \frac{\text{Cents}}{\text{Carrot}}\right]\) as

\[+35 \text{ Cents}\]
1. **Question:** Given the problem in **Dollars**

\[ x > +341.17 \]

what is the *graph* of its solution subset?

**Discussion:** This inequation lets IN all the numbers that are *larger* than +341.17.

The inequation is *strict* so that it leaves OUT the *boundary point* +341.17.

The *graph* of the solution subset is therefore:

![Graph of solution subset](image1)

2. **Question:** Given the problem in **Dollars**

\[ x \geq -152.78 \]

what is the *graph* of its solution subset?

**Discussion:** This inequation lets IN all the numbers that are *larger* than −152.78.

The inequation is *lenient* so that it lets IN the *boundary point* −152.78.

The *graph* of the solution subset is therefore:

![Graph of solution subset](image2)

3. **Question:** Given the problem in **Dollars**

\[ x < -371.45 \]

what is the *graph* of its solution subset?

**Discussion:** This inequation lets IN all the numbers that are *smaller* than −371.45.

The inequation is *strict* so that it leaves OUT the *boundary point* −371.45.

The *graph* of the solution subset is therefore:
4. **Question:** Given the problem in Dollars

\[ x \leq +713.66 \]

what is the *graph* of its solution subset?

**Discussion:** This inequation lets IN all the numbers that are *smaller* than +713.66.

The inequation is *strict* so that it lets IN the *boundary point* +713.66.

The *graph* of the solution subset is therefore:

5. **Question:** Given the problem in Dollars

\[ x \neq +451.89 \]

what is the *graph* of its solution subset?

**Discussion:** This inequation lets IN all the numbers that are *different* from +451.89.

The inequation is *total* so that it leaves OUT the *boundary point* +451.89.

The *graph* of the solution subset is therefore:

6. **Question:** Given the problem in Dollars

\[ x = +832.91 \]

what is the *graph* of its solution subset?

**Discussion:** This inequation lets IN only the number +832.91.

The *graph* of the solution subset is therefore:
7. **Question:** Given the problem in Dollars

\[ x \oplus +7 > +5 \]

what is the graph of its solution set?

**Discussion:**

i. To locate the boundary of the solution subset, we must solve the associated equation.

   i. The associated equation is

   \[ x \oplus +7 = +5 \]

   ii. We use the Reduction Approach to solve the associated equation, that is we “ominus” both sides with \(-7\), that is we “oplus” both sides with the opposite of \(-7\):

   \[ x \oplus +7 \oplus -7 = +5 \oplus -7 \]

   which boils down to the basic equation in Dollars

   \[ x = -2 \]

   which the Fairness Theorem ensures to be equivalent to the original signed translation equation in Dollars

   \[ x \oplus +7 = +5 \]

   which therefore has the solution of the basic equation, \(-2\), as its own solution.

   iii. So \(-2\) is the boundary point of the solution subset of the problem but since the inequation involves the verb \(>\), it is strict and the boundary point \(-2\) is non-included in the solution subset of the problem.

   iv. The graph of the boundary of the solution subset of the problem is therefore:

   ![Graph of boundary solution subset]

ii. To locate the interior of the solution subset, we use the Pasch Procedure:
i. The boundary point splits the data set in two regions which we label Section A and Section B as follows:

ii. We test Section A with, for instance, $-100$ and, since

$$x \oplus +7 > +5|_{x=-100} \text{ is FALSE}$$

we get that $-100$ is a non-solution of the inequation in Dollars

$$x \oplus +7 > +5$$

and Pasch’s Theorem then tells us that all number-phrases in Section A are non-included in the solution subset.

iii. We test Section B with, for instance, $+100$ and, since

$$x \oplus +7 > +5|_{x=+100} \text{ is TRUE}$$

we get that $+100$ is a solution of the inequation in Dollars

$$x \oplus +7 > +5$$

and Pasch’s Theorem then tells us that all number-phrases in Section B are included in the solution subset.

iv. The graph of the solution subset of the problem is therefore:

8. Question: Given the problem in Dollars

$$-4 \odot x > -12$$

what is the graph of its solution set?

Discussion:

i. To locate the boundary of the solution subset, we must solve the associated equation.

i. The associated equation is

$$-4 \odot x = -12$$
ii. We use the Reduction Approach to solve the associated equation, that is we “odiv” both sides with $-4$:

$$-4 \otimes x \odiv -4 = -12 \otimes -4$$

which boils down to the basic equation in Dollars

$$x = +3$$

which the Fairness Theorem ensures to be equivalent to the original signed dilation equation in Dollars

$$-4 \otimes x = -12$$

which therefore has the solution of the basic equation, +3, as its own solution.

iii. So +3 is the boundary point of the solution subset of the problem but since the inequation involves the verb $>$, it is strict and the boundary point +3 is non-included in the solution subset of the problem.

iv. The graph of the boundary of the solution subset of the problem is therefore:

$$\begin{array}{c}
\text{Section A} \\
\text{Section B}
\end{array}$$

ii. To locate the interior of the solution subset, we use the Pasch Procedure:

i. The boundary point splits the data set in two regions which we label Section A and Section B as follows:

$$\begin{array}{c}
\text{Section A} \\
\text{Section B}
\end{array}$$

ii. We test Section A with, for instance, $-100$ and, since

$$-4 \otimes x > -12\big|_{x=-100} \text{ is TRUE}$$

we get that $-100$ is a solution of the inequation in Dollars

$$-4 \otimes x > -12$$

and Pasch’s Theorem then tells us that all number-phrases in Section A are included in the solution subset.

iii. We test Section B with, for instance, $+100$ and, since

$$-4 \otimes x > -12\big|_{x=+100} \text{ is FALSE}$$
we get that +100 is a non-solution of the inequation in Dollars
\[-4 \otimes x > -12\]
and Pasch’s Theorem then tells us that all number-phrases in Section B are non-included in the solution subset.

iv. The graph of the solution subset of the problem is therefore:

9. Question: Given the problem in Dollars

\[+3x + 7 > -5x - 9\]

what is the graph of its solution set?

Discussion:

i. We solve the associated equation:

\[+3x + 7 = -5x - 9\]

Our ultimate goal is to reduce to a basic equation:

\[x = \text{numerator}\]

So, \(+7\) must go away from the left side and \(-5x\) must go away from the right side:

\[+3x + 7 = -5x - 9\]

• To get rid of \(+7\) we add the opposite of \(+7\) to the left side and, to be fair, also to the right side:

\[+3x + 7 = -5x - 9\]

\[-7\]

which gives us

\[+3x = -5x - 16\]
To get rid of $-5x$ we add the opposite of $-5x$ to the right side and, to be fair, also to the left side:

\[
\begin{align*}
+3x & = -5x - 16 \\
+5x & +5x
\end{align*}
\]

which gives us

\[
+8x = -16
\]

So, altogether, the associated equation has been separated down to the following basic equation

\[
x = -2
\]

ii. The boundary point $-2$ is not a solution since the given inequation is strict and $-2$ is therefore non-included in the solution subset.

iii. To locate the interior of the solution subset, we use the PASCH Procedure

i. The boundary point $-2$ divides the interior of the solution subset in two sections which we shall call Section A and Section B.

\[
\begin{align*}
\text{Section A} & \quad \text{Section B} \\
\end{align*}
\]

ii. We test Section A with, for instance, $-1000$:

\[
\begin{align*}
+3x + 7|_{x=-1000} & > -5x - 9|_{x=-1000} \\
+3(-1000) + 7 & > -5(-1000) - 9 \\
-3000 + 7 & > +5000 - 9 \\
-2993 & > +4991
\end{align*}
\]

Since this is FALSE, we have that $-1000$ is a non-solution of the inequation and then THEOREM #4 tells us that all number-phrases in Section A are non-included in the solution subset.
iii. We test Section B with, for instance, $+1000$

$$\begin{align*}
+3x + 7|_{x=+1000} &> -5x - 9|_{x=+1000} \\
+3(+1000) + 7 &> -5(+1000) - 9 \\
+3000 + 7 &> -5000 - 9 \\
+3007 &> -5009
\end{align*}$$

Since this is TRUE, we have that $+1000$ is a solution of the inequation and then THEOREM #4 tells us that all number-phrases in Section B are included in the solution subset.

10. Question: Given the problem in Dollars

$$-6x - 5 < +4x + 25$$

what is the graph of its solution set?

Discussion:

i. We solve the associated equation:

$$-6x - 5 = +4x + 25$$

Our ultimate goal is to separate as follows:

$$x’s = \text{number}$$

So, $-5$ must go away from the left side and $+4x$ must go away from the right side:

$$-6x \underline{-5} = +4x + 25$$

- To get rid of $-5$ we add the opposite of $-5$ to the left side and, to be fair, also to the right side:

$$-6x \underline{-5} = +4x + 25$$
which gives us
\[ -6x = +4x + 30 \]

*To get rid of +4x we add the opposite of +4x to the right side and, to be fair, also to the left side:

\[
\begin{align*}
-6x & = +4x + 30 \\
-4x & = -4x
\end{align*}
\]

which gives us
\[ -10x = +30 \]

So, altogether, the associated equation has been separated down to the following basic equation
\[ x = -3 \]

**ii.** The boundary point \(-3\) is not a solution since the given inequation is total and \(-3\) is therefore non-included in the solution subset

**iii.** To locate the interior of the solution subset, we use the PASCH Procedure

1. The boundary point \(-3\) splits the interior of the solution subset in two sections which we shall call Section A and Section B.

2. We test Section A with, for instance, \(-1000\):

\[
\begin{align*}
-6x - 5|_{x=-1000} & < +4x + 25|_{x=-1000} \\
-6(-1000) - 5 & < +4(-1000) + 25 \\
+6000 - 5 & < -4000 + 25
\end{align*}
\]
Since this is FALSE, we have that $-1000$ is a non-solution of the inequality and then THEOREM #4 tells us that all number-phrases in Section A are non-included in the solution subset.

\[ +5995 < -3975 \]

\( ii. \) We test Section B with, for instance, $+1000$

\[ -6x - 5|_{x=+1000} < +4x + 25|_{x=+1000} \]
\[ -6(+1000) - 5 < +4(+1000) + 25 \]
\[ -6000 - 5 < +4000 + 25 \]
\[ -6005 < +4025 \]

Since this is TRUE, we have that $+1000$ is a solution of the inequation and then THEOREM #4 tells us that all number-phrases in Section B are included in the solution subset.

11. Question: Given the problem in Dollars

\[-4x + 7 \leq +6x - 23\]

what is the graph of its solution set?

Discussion:

i. We solve the associated equation:

\[-4x + 7 = +6x - 23\]

Our ultimate goal is to separate into a basic equation as follows:

\[ x = \text{numerator} \]

So, $+7$ must go away from the left side and $+6x$ must go away from the right side:

\[-4x + 7 = +6x - 23\]
To get rid of $+7$ we add the opposite of $+7$ to the left side and, to be fair, also to the right side:

\[-4x + 7 = +6x - 23\]

which gives us

\[-4x = +6x - 30\]

To get rid of $+6x$ we add the opposite of $+6x$ to the right side and, to be fair, also to the left side:

\[-4x = +6x - 30\]

which gives us

\[-10x = -16\]

So, altogether, the associated equation has been separated down to the following basic equation

\[x = +1.6\]

ii. The boundary point $+1.6$ is a solution since the given inequation is lenient and $+1.6$ is therefore included in the solution subset

iii. To locate the interior of the solution subset we use the Pasch Procedure

\[\text{Section A} \quad \text{Section B}\]

\[\text{Dollars}\]
ii. We test Section A with, for instance, $-1000$:

$$\begin{align*}
-4x + 7|_{x=-1000} &> +6x - 23|_{x=-1000} \\
-4(-1000) + 7 &\leq +6(-1000) - 23 \\
+4000 + 7 &\leq -6000 - 23 \\
+4007 &\leq -6023
\end{align*}$$

Since this is FALSE, we have that $-1000$ is a non-solution of the inequality and then THEOREM #4 tells us that all number-phrases in Section A are non-included in the solution subset.

\[\text{Section A} \quad \bullet \quad \text{Section B} \quad \text{Dollars}\]

\[\text{boundary}\]

iii. We test Section B with, for instance, $+1000$

$$\begin{align*}
-4x + 7|_{x=+1000} &> +6x - 23|_{x=+1000} \\
-4(+1000) + 7 &> +6(+1000) - 23 \\
-4000 + 7 &> +6000 - 23 \\
-3993 &> +5977
\end{align*}$$

Since this is TRUE, we have that $+1000$ is a solution of the inequality and then THEOREM #4 tells us that all number-phrases in Section B are included in the solution subset.

\[\text{Section A} \quad \bullet \quad \text{Section B} \quad \text{Dollars}\]

\[\text{boundary}\]

12. Question: Given the double basic problem in Dollars

$$\begin{align*}
\text{both} \quad \begin{cases} x > -337.41 \\ x \leq +568.92 \end{cases}
\end{align*}$$

what is the graph of its solution subset?

Discussion:

I. To locate the boundary of the solution subset of the double problem, we must solve the associated problem.

i. The associated problem in Dollars is:
ii. We solve each one of the two associated equations.

a. The solution of the first associated equation is $-337.41$ so $-337.41$ is a boundary point of the solution subset of the double problem.

b. The solution of the second associated equation is $+568.92$ so $+568.92$ is a boundary point of the solution subset of the double problem.

iii. We check each boundary points to find if it is included or non-included in the solution subset of the double problem.

a. We check the boundary point $-337.41$. Since

\[
\text{BOTH } \begin{cases} x > -337.41 \\ x \leq +568.92 \end{cases} \quad x = -337.41
\]

is false

the boundary point $-337.41$ is non-included in the solution subset of the double problem.

b. We check the boundary point $+568.92$. Since

\[
\text{BOTH } \begin{cases} x > -337.41 \\ x \leq +568.92 \end{cases} \quad x = +568.92
\]

is false

the boundary point $+568.92$ is non-included in the solution subset of the double problem.

iv. The graph of the boundary of the solution subset of the double problem is therefore:

\[\text{Dollars} \quad -337.41 \quad +568.92\]

II. To locate the interior of the solution subset of the double problem, we use the PASCH PROCEDURE:

i. The boundary points split the data set in three regions which we label Section A, Section B and Section C as follows:

<table>
<thead>
<tr>
<th>Section A</th>
<th>Section B</th>
<th>Section C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-337.41$</td>
<td></td>
<td>$+568.92$</td>
</tr>
<tr>
<td>$\text{Dollars}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[\text{Dollars} \quad -337.41 \quad +568.92\]
ii. We test Section A with, for instance, the point $-1000$. Since
\[
\text{BOTH } \begin{cases} x > -337.41 \\ x \leq +568.92 \end{cases} \bigg|_{x=-1000} \quad \text{is FALSE}
\]
the point $-1000$ is a non-solution of the double problem in Dollars
\[
\text{BOTH } \begin{cases} x > -337.41 \\ x \leq +568.92 \end{cases}
\]
and Pasch’s Theorem then tells us that all points in Section A are non-included in the solution subset of the double problem.

iii. We test Section B with, for instance, 0 and, since
\[
\text{BOTH } \begin{cases} x > -337.41 \\ x \leq +568.92 \end{cases} \bigg|_{x=0} \quad \text{is TRUE}
\]
the point 0 is a solution of the double problem in Dollars
\[
\text{BOTH } \begin{cases} x > -337.41 \\ x \leq +568.92 \end{cases}
\]
and Pasch’s Theorem then tells us that all number-phrases in Section B are included in the solution subset of the double problem.

iv. We test Section C with, for instance, $+1000$ and, since
\[
\text{BOTH } \begin{cases} x > -337.41 \\ x \leq +568.92 \end{cases} \bigg|_{x=+1000} \quad \text{is FALSE}
\]
the point $+1000$ is a non-solution of the double problem in Dollars
\[
\text{BOTH } \begin{cases} x > -337.41 \\ x \leq +568.92 \end{cases}
\]
and Pasch’s Theorem then tells us that all number-phrases in Section C are non-included in the solution subset of the double problem.

III. Altogether, the graph of the solution subset of the double problem in Dollars
\[
\text{BOTH } \begin{cases} x > -337.41 \\ x \leq +568.92 \end{cases}
\]
is therefore:
13. **Question:** Given the double basic problem in **Dollars**

\[
\begin{align*}
&\text{BOTH} \ \\
&\left\{ \begin{array}{l}
x \geq +629.51 \\
x \leq +268.92
\end{array} \right.
\end{align*}
\]

what is the *graph* of its solution subset?

**Discussion:**

**I.** To locate the *boundary* of the solution subset of the double problem, we must solve the *associated problem.*

   **i.** The *associated problem* in **Dollars** is:

   \[
   \begin{align*}
   &\text{OR} \ \\
   &\left\{ \begin{array}{l}
x = +629.51 \\
x = +268.92
\end{array} \right.
\end{align*}
\]

   **ii.** We solve each one of the two *associated equations.*

   **a.** The solution of the first associated equation is +629.51 so +629.51 is a *boundary point* of the solution subset of the double problem.

   **b.** The solution of the second associated equation is +268.92 so +268.92 is a *boundary point* of the solution subset of the double problem.

   **iii.** We check each boundary points to find if it is *included* or *non-included* in the solution subset of the double problem.

   **a.** We check the boundary pointt +629.51. Since

   \[
   \begin{align*}
   &\text{BOTH} \ \\
   &\left\{ \begin{array}{l}
x \geq +629.51 \\
x \leq +268.92
\end{array} \right. \\
   &x = +629.51
\end{align*}
\]

   is FALSE

   the boundary point +629.51 is *non-included* in the solution subset of the double problem.

   **b.** We check the boundary pointt +268.92. Since

   \[
   \begin{align*}
   &\text{BOTH} \ \\
   &\left\{ \begin{array}{l}
x \geq +629.51 \\
x \leq +268.92
\end{array} \right. \\
   &x = +268.92
\end{align*}
\]

   is FALSE
the boundary point $+268.92$ is *non-included* in the solution subset of the double problem.

**iv.** The graph of the *boundary* of the solution subset of the double problem is therefore:

![Graph of boundary points]

**II.** To locate the *interior* of the solution subset of the double problem, we use the *Pasch Procedure*:

**i.** The boundary points split the data set in three regions which we label Section A, Section B and Section C as follows:

![Graph of section A, B, C]

**ii.** We test Section A with, for instance, the point $-1000$. Since

\[
\text{BOTH } \begin{cases} 
  x \geq +629.51 \\
  x \leq +268.92 
\end{cases} \big|_{x=-1000} \quad \text{is FALSE}
\]

the point $-1000$ is a *non-solution* of the double problem in *Dollars*

\[
\text{BOTH } \begin{cases} 
  x \geq +629.51 \\
  x \leq +268.92 
\end{cases}
\]

and *Pasch’s Theorem* then tells us that *all* points in Section A are *non-included* in the solution subset of the double problem.

**iii.** We test Section B with, for instance, the point $0$. Since

\[
\text{BOTH } \begin{cases} 
  x \geq +629.51 \\
  x \leq +268.92 
\end{cases} \big|_{x=0} \quad \text{is FALSE}
\]

we get that $0$ is a *solution* of the double problem in *Dollars*

\[
\text{BOTH } \begin{cases} 
  x \geq +629.51 \\
  x \leq +268.92 
\end{cases}
\]

and *Pasch’s Theorem* then tells us that *all* points in Section B are *non-included* in the solution subset of the double problem.
iv. We test Section C with, for instance, the point +1000. Since

$$\text{BOTH } \begin{cases} x \geq 629.51 \\ x \leq 268.92 \end{cases} \text{ is FALSE}$$

we get that +1000 is a non-solution of the double problem in Dollars

$$\text{BOTH } \begin{cases} x \geq 629.51 \\ x \leq 268.92 \end{cases}$$

and Pasch’s Theorem then tells us that all points in Section C are non-included in the solution subset of the double problem.

III. Altogether, the graph of the solution subset of the double problem in Dollars

$$\text{BOTH } \begin{cases} x \geq 629.51 \\ x \leq 268.92 \end{cases}$$

is therefore:

\[\text{Graph of solution subset of double problem}\]

\[\text{Dollars}\]

14. Question: Given the double basic problem in Dollars

$$\text{BOTH } \begin{cases} x \geq 391.51 \\ x \leq 391.51 \end{cases}$$

what is the graph of its solution set?

Discussion:

I. To locate the boundary of the solution subset of the double problem, we must solve the associated problem.

i. The associated problem in Dollars is:

$$\text{OR } \begin{cases} x = 391.51 \\ x = 391.51 \end{cases}$$

ii. The two associated equations are the same. The solution is +391.51. So +391.51 is the only boundary point of the solution subset of the double problem.
iii. We check the boundary point +391.51 to find if it is included or non-included in the solution subset of the double problem. Since

\[
\text{BOTH} \left\{ \begin{array}{l}
    x \geq +391.51 \\
    x \leq +391.51
\end{array} \right\}_{x = +391.51} \text{ is TRUE}
\]

the boundary point +391.51 is included in the solution subset of the double problem.

iv. The graph of the boundary of the solution subset of the double problem is therefore:

\[
\begin{array}{c}
\text{Section A} \\
\text{Section B}
\end{array}
\]

II. To locate the interior of the solution subset of the double problem, we use the PASCH PROCEDURE:

i. The boundary point splits the data set in two regions which we label Section A and Section B as follows:

\[
\begin{array}{c}
\text{Section A} \\
\text{Section B}
\end{array}
\]

ii. We test Section A with, for instance, the point −1000. Since

\[
\text{BOTH} \left\{ \begin{array}{l}
    x \geq +391.51 \\
    x \leq +391.51
\end{array} \right\}_{x = -1000} \text{ is FALSE}
\]

the point −1000 is a non-solution of the double problem in Dollars

\[
\text{BOTH} \left\{ \begin{array}{l}
    x \geq +391.51 \\
    x \leq +391.51
\end{array} \right\}
\]

and Pasch’s Theorem then tells us that all points in Section A are non-included in the solution subset of the double problem.

iii. We test Section B with, for instance, the point +1000. Since

\[
\text{BOTH} \left\{ \begin{array}{l}
    x \geq +391.51 \\
    x \leq +391.51
\end{array} \right\}_{x = +1000} \text{ is FALSE}
\]
the point +1000 is a non-solution of the double problem in Dollars

\[
\begin{align*}
\text{BOTH} & \cases{x \geq +391.51 \\
x \leq +391.51}
\end{align*}
\]

and Pasch’s Theorem then tells us that all points in Section B are non-included in the solution subset of the double problem.

**III.** Altogether, the graph of the solution subset of the double problem in Dollars

\[
\begin{align*}
\text{BOTH} & \cases{x \geq +391.51 \\
x \leq +391.51}
\end{align*}
\]

is therefore:

15. **Question:** Given the double basic problem in Dollars

\[
\begin{align*}
\text{BOTH} & \cases{x > +911.52 \\
x < +911.52}
\end{align*}
\]

what is the graph of its solution set?

**Discussion:**

**I.** To locate the boundary of the solution subset of the double problem, we must solve the associated problem.

i. The associated problem in Dollars is:

\[
\begin{align*}
\text{OR} & \cases{x = +911.52 \\
x = +911.52}
\end{align*}
\]

ii. The two associated equations are the same. The solution is +911.52. So +911.52 is the only boundary point of the solution subset of the double problem.

iii. We check the boundary point +911.52 to find if it is included or non-included in the solution subset of the double problem.

Since

\[
\begin{align*}
\text{BOTH} & \cases{x > +911.52 \\
x < +911.52} \\
\text{at} & \quad x = +911.52
\end{align*}
\]

is false
the boundary point +911.52 is non-included in the solution subset of the double problem.

iv. The graph of the boundary of the solution subset of the double problem is therefore:

II. To locate the interior of the solution subset of the double problem, we use the Pasch Procedure:

i. The boundary point splits the data set in two regions which we label Section A and Section B as follows:

ii. We test Section A with, for instance, the point −1000. Since

\[ \text{BOTH} \begin{cases} x > +911.52 \\ x < +911.52 \end{cases} \quad \text{is FALSE} \]

the point −1000 is a non-solution of the double problem in Dollars

\[ \text{BOTH} \begin{cases} x > +911.52 \\ x < +911.52 \end{cases} \]

and Pasch’s Theorem then tells us that all points in Section A are non-included in the solution subset of the double problem.

iii. We test Section B with, for instance, the point +1000. Since

\[ \text{BOTH} \begin{cases} x > +911.52 \\ x < +911.52 \end{cases} \quad \text{is FALSE} \]

the point +1000 is a non-solution of the double problem in Dollars

\[ \text{BOTH} \begin{cases} x > +911.52 \\ x < +911.52 \end{cases} \]

and Pasch’s Theorem then tells us that all points in Section B are non-included in the solution subset of the double problem.
III. Altogether, the graph of the solution subset of the double problem in Dollars

\[
\begin{align*}
\text{BOTH} & \quad \begin{cases} 
  x > +911.52 \\
  x < +911.52 
\end{cases}
\end{align*}
\]

is therefore:

-------------------------------------
| Dollars |
-------------------------------------

16. Question: Given the double basic problem in Dollars

\[
\begin{align*}
\text{BOTH} & \quad \begin{cases} 
  x \neq -786.33 \\
  x \geq +315.32 
\end{cases}
\end{align*}
\]

what is the graph of its solution set?

Discussion:

I. To locate the boundary of the solution subset of the double problem, we must solve the associated problem.

   i. The associated problem in Dollars is:

   \[
   \begin{align*}
   \text{OR} & \quad \begin{cases} 
   x = -786.33 \\
   x = +315.32 
\end{cases}
   \end{align*}
   \]

   ii. We solve each one of the two associated equations.

a. The solution of the first associated equation is $-786.33$. So $-786.33$ is a boundary point of the solution subset of the double problem.

b. The solution of the second associated equation is $+315.32$. So $+315.32$ is a boundary point of the solution subset of the double problem.

   iii. We check each boundary points to find if it is included or non-included in the solution subset of the double problem.

a. We check the boundary point $-786.33$. Since

\[
\begin{align*}
\text{BOTH} & \quad \begin{cases} 
   x \neq -786.33 \\
   x \geq +315.32 
\end{cases} \quad \text{is FALSE}
\end{align*}
\]

the boundary point $-786.33$ is non-included in the solution subset of the double problem.
b. We check the boundary point $+315.32$. Since

\[
\text{BOTH } \begin{cases} x \neq -786.33 \\ x \geq +315.32 \end{cases} \quad \text{is TRUE}
\]

the boundary point $+315.32$ is included in the solution subset of the double problem.

\[\text{\textbf{iv. The graph of the boundary of the solution subset of the double problem is therefore:}}\]

\[\text{\textbf{II. To locate the interior of the solution subset of the double problem, we use the \textsc{Pasch Procedure}:}}\]

i. The boundary points split the data set in three regions which we label Section A, Section B and Section C as follows:

\[\text{\textbf{\begin{align*}
\text{Section A} & \quad \text{Section B} \quad \text{Section C} \\
\begin{array}{c}
-786.33 \\
+315.32
\end{array}
\end{align*}}\]

\[\text{\textbf{\textbf{ii. We test Section A with, for instance, the point } -1000. Since}}\]

\[
\text{BOTH } \begin{cases} x \neq -786.33 \\ x \geq +315.32 \end{cases} \quad \text{is FALSE}
\]

the point $-1000$ is a non-solution of the double problem in Dollars and \textsc{Pasch’s Theorem} then tells us that all points in Section A are non-included in the solution subset of the double problem.

\[\text{\textbf{\textbf{iii. We test Section B with, for instance, the point } 0. Since}}\]

\[
\text{BOTH } \begin{cases} x \neq -786.33 \\ x \geq +315.32 \end{cases} \quad \text{is FALSE}
\]

the point $0$ is a non-solution of the double problem in Dollars.
and Pasch’s Theorem then tells us that all points in Section B are non-included in the solution subset of the double problem.

iv. We test Section C with, for instance, the point +1000. Since

\[
\text{both } \begin{cases} x \neq -786.33 \\ x \geq +315.32 \end{cases} \text{ for } x = +1000
\]

the point +1000 is a solution of the double problem in Dollars

\[
\text{both } \begin{cases} x \neq -786.33 \\ x \geq +315.32 \end{cases}
\]

and Pasch’s Theorem then tells us that all points in Section C are included in the solution subset of the double problem.

III. Altogether, the graph of the solution subset of the double problem in Dollars

\[
\text{both } \begin{cases} x \neq -786.33 \\ x \geq +315.32 \end{cases}
\]

is therefore:

17. Question: Given the double basic problem in Dollars

\[
\text{both } \begin{cases} x \leq +315.32 \\ x \neq +272.81 \end{cases}
\]

what is the graph of its solution set?

Discussion:

I. To locate the boundary of the solution subset of the double problem, we must solve the associated problem.

i. The associated problem in Dollars is:

\[
\text{OR } \begin{cases} x = +315.32 \\ x = +272.81 \end{cases}
\]
ii. We solve each one of the two associated equations.

a. The solution of the first associated equation is $+315.32$ so $+315.32$ is a boundary point of the solution subset of the double problem.

b. The solution of the second associated equation is $+272.81$ so $+272.81$ is a boundary point of the solution subset of the double problem.

iii. We check each boundary point to find if it is included or non-included in the solution subset of the double problem.

a. We check the boundary point $+315.32$. Since

$$\text{both } \begin{cases} x \leq +315.32 \\ x \neq +272.81 \end{cases} \text{ is TRUE}$$

the boundary point $+315.32$ is included in the solution subset of the double problem.

b. We check the boundary point $+272.81$. Since

$$\text{both } \begin{cases} x \leq +315.32 \\ x \neq +272.81 \end{cases} \text{ is FALSE}$$

the boundary point $+272.81$ is non-included in the solution subset of the double problem.

iv. The graph of the boundary of the solution subset of the double problem is therefore:

\[\text{Dollars}\]

\[\begin{array}{c}
-272.81 \\
+315.32
\end{array}\]

\[\text{Dollars}\]

II. To locate the interior of the solution subset of the double problem, we use the PASCH PROCEDURE:

i. The boundary points split the data set in three regions which we label Section A, Section B and Section C as follows:

\[\text{Dollars}\]

\[\begin{array}{c}
-272.81 \\
+315.32
\end{array}\]

\[\text{Dollars}\]

ii. We test Section A with, for instance, the point $-1000$. Since
the point $-1000$ is a solution of the double problem in Dollars

both \[ \begin{cases} x \leq +315.32 \\ x \neq +272.81 \end{cases} \] \[ x = -1000 \]
is TRUE

and Pasch’s Theorem then tells us that all points in Section A are included in the solution subset of the double problem.

iii. We test Section B with, for instance, the point 0. Since

both \[ \begin{cases} x \leq +315.32 \\ x \neq +272.81 \end{cases} \] \[ x = 0 \]
is TRUE

the point 0 is a solution of the double problem in Dollars

both \[ \begin{cases} x \leq +315.32 \\ x \neq +272.81 \end{cases} \]

and Pasch’s Theorem then tells us that all points in Section B are included in the solution subset of the double problem.

iv. We test Section C with, for instance, the point +1000. Since

both \[ \begin{cases} x \leq +315.32 \\ x \neq +272.81 \end{cases} \] \[ x = +1000 \]
is FALSE

the point +1000 is a non-solution of the double problem in Dollars

both \[ \begin{cases} x \leq +315.32 \\ x \neq +272.81 \end{cases} \]

and Pasch’s Theorem then tells us that all points in Section C are non-included in the solution subset of the double problem.

III. Altogether, the graph of the solution subset of the double problem in Dollars

both \[ \begin{cases} x \leq +315.32 \\ x \neq +272.81 \end{cases} \]
is therefore:

\[ \text{Dollars} \]

\[ +272.81 \]

\[ +315.32 \]
18. Question: Given the double basic problem in Dollars

\[
\begin{aligned}
\text{BOTH} & \quad \begin{cases} 
  x = +786.33 \\
  x \geq +222.91 
\end{cases} \\
\end{aligned}
\]

what is the graph of its solution set?

Discussion:

I. To locate the boundary of the solution subset of the double problem, we must solve the associated problem.

i. The associated problem in Dollars is:

OR \[
\begin{align*}
  x &= +786.33 \\
  x &= +222.91 
\end{align*}
\]

ii. We solve each one of the two associated equations.

a. The solution of the first associated equation is +786.33 so +786.33 is a boundary point of the solution subset of the double problem.

b. The solution of the second associated equation is +222.91 so +222.91 is a boundary point of the solution subset of the double problem.

iii. We check each boundary points to find if it is included or non-included in the solution subset of the double problem.

a. We check the boundary pointt +786.33. Since

\[
\begin{aligned}
\text{BOTH} & \quad \begin{cases} 
  x = +786.33 \\
  x \geq +222.91 
\end{cases} \\
\end{aligned}
\]

the boundary point +786.33 is included in the solution subset of the double problem.

b. We check the boundary pointt +222.91. Since

\[
\begin{aligned}
\text{BOTH} & \quad \begin{cases} 
  x = +786.33 \\
  x \geq +222.91 
\end{cases} \\
\end{aligned}
\]

the boundary point +222.91 is non-included in the solution subset of the double problem.

iv. The graph of the boundary of the solution subset of the double problem is therefore:
II. To locate the *interior* of the solution subset of the double problem, we use the Pasch Procedure:

i. The boundary points split the data set in three regions which we label Section A, Section B and Section C as follows:

<table>
<thead>
<tr>
<th>Section A</th>
<th>Section B</th>
<th>Section C</th>
</tr>
</thead>
<tbody>
<tr>
<td>(+222.91)</td>
<td>(+786.33)</td>
<td>(+786.33)</td>
</tr>
</tbody>
</table>

ii. We test Section A with, for instance, the point \(-1000\). Since

\[
\begin{align*}
\text{BOTH} \quad & \quad \left\{ \begin{array}{l}
x = +786.33 \\
x \geq +222.91 \\
x := -1000
\end{array} \right. \\
\text{is FALSE}
\end{align*}
\]

the point \(-1000\) is a *non-solution* of the double problem in Dollars

\[
\text{BOTH} \quad \left\{ \begin{array}{l}
x = +786.33 \\
x \geq +222.91
\end{array} \right. 
\]

and **Pasch’s Theorem** then tells us that *all* points in Section A are *non-included* in the solution subset of the double problem.

iii. We test Section B with, for instance, the point 0. Since

\[
\begin{align*}
\text{BOTH} \quad & \quad \left\{ \begin{array}{l}
x = +786.33 \\
x \geq +222.91 \\
x := 0
\end{array} \right. \\
\text{is TRUE}
\end{align*}
\]

the point 0 is a *solution* of the double problem in Dollars

\[
\text{BOTH} \quad \left\{ \begin{array}{l}
x = +786.33 \\
x \geq +222.91
\end{array} \right. 
\]

and **Pasch’s Theorem** then tells us that *all* points in Section B are *included* in the solution subset of the double problem.

iv. We test Section C with, for instance, the point +1000. Since

\[
\begin{align*}
\text{BOTH} \quad & \quad \left\{ \begin{array}{l}
x = +786.33 \\
x \geq +222.91 \\
x := +1000
\end{array} \right. \\
\text{is FALSE}
\end{align*}
\]
the point +1000 is a non-solution of the double problem in Dollars

\[
\begin{align*}
\text{both} & \quad \begin{cases} 
  x = +786.33 \\
  x \geq +222.91
\end{cases}
\end{align*}
\]

and Pasch’s Theorem then tells us that all points in Section C are non-included in the solution subset of the double problem.

III. Altogether, the graph of the solution subset of the double problem in Dollars

\[
\begin{align*}
\text{both} & \quad \begin{cases} 
  x = +786.33 \\
  x \geq +222.91
\end{cases}
\end{align*}
\]

is therefore:

19. Question: Given the double basic problem in Dollars

\[
\begin{align*}
\text{either one or both} & \quad \begin{cases} 
  x = -281.88 \\
  x > +876.33
\end{cases}
\end{align*}
\]

what is the graph of its solution subset?

Discussion:

I. To locate the boundary of the solution subset of the double problem, we must solve the associated problem.

i. The associated problem in Dollars is:

\[
\begin{align*}
\text{or} & \quad \begin{cases} 
  x = -281.88 \\
  x = +876.33
\end{cases}
\end{align*}
\]

ii. We solve each one of the two associated equations.

a. The solution of the first associated equation is -281.88. So -281.88 is a boundary point of the solution subset of the double problem.

b. The solution of the second associated equation is +876.33. So +876.33 is a boundary point of the solution subset of the double problem.

iii. We check each boundary points to find if it is included or non-included in the solution subset of the double problem.
a. We check the boundary point $-281.88$. Since

$$\text{either one or both } \begin{cases} x = -281.88 \\ x > +876.33 \end{cases}_{x=-281.88} \text{ is TRUE}$$

the boundary point $-281.88$ is included in the solution subset of the double problem.

b. We check the boundary point $+876.33$. Since

$$\text{either one or both } \begin{cases} x = -281.88 \\ x > +876.33 \end{cases}_{x=+876.33} \text{ is FALSE}$$

the boundary point $+876.33$ is non-included in the solution subset of the double problem.

iv. The graph of the boundary of the solution subset of the double problem is therefore:

![Graph of boundary subset]

II. To locate the interior of the solution subset of the double problem, we use the Pasch Procedure:

i. The boundary points split the data set in three regions which we label Section A, Section B and Section C as follows:

![Regions A, B, and C]

ii. We test Section A with, for instance, the point $-1000$. Since

$$\text{either one or both } \begin{cases} x = -281.88 \\ x > +876.33 \end{cases}_{x=-1000} \text{ is FALSE}$$

the point $-1000$ is a non-solution of the double problem in Dollars

$$\text{either one or both } \begin{cases} x = -281.88 \\ x > +876.33 \end{cases}$$

and Pasch’s Theorem then tells us that all points in Section A are non-included in the solution subset of the double problem.
iii. We test Section Y with, for instance, the point 0. Since

\[
\text{EITHER ONE OR BOTH } \begin{cases} 
  x = -281.88 \\
  x > +876.33 \\
\end{cases} \quad x := 0
\]

the point 0 is a non-solution of the double problem in Dollars

\[
\text{EITHER ONE OR BOTH } \begin{cases} 
  x = -281.88 \\
  x > +876.33 \\
\end{cases}
\]

and Pasch’s Theorem then tells us that all points in Section B are non-included in the solution subset of the double problem.

iv. We test Section C with, for instance, the point +1000. Since

\[
\text{EITHER ONE OR BOTH } \begin{cases} 
  x = -281.88 \\
  x > +876.33 \\
\end{cases} \quad x := +1000
\]

the point +1000 is a solution of the double problem in Dollars

\[
\text{EITHER ONE OR BOTH } \begin{cases} 
  x = -281.88 \\
  x > +876.33 \\
\end{cases}
\]

and Pasch’s Theorem then tells us that all points in Section C are included in the solution subset of the double problem.

III. Altogether, the graph of the solution subset of the double problem in Dollars

\[
\text{EITHER ONE OR BOTH } \begin{cases} 
  x = -281.88 \\
  x > +876.33 \\
\end{cases}
\]

is therefore:

![Graph of solution subset](image)

20. Question: Given the double basic problem in Dollars

\[
\text{EITHER ONE OR BOTH } \begin{cases} 
  x \neq -786.33 \\
  x \geq +315.32 \\
\end{cases}
\]

what is the graph of its solution set?

Discussion:
I. To locate the boundary of the solution subset of the double problem, we must solve the associated problem.

i. The associated problem in Dollars is:

\[
\begin{align*}
\text{OR } & \begin{cases} 
  x = -786.33 \\
  x = +315.32 
\end{cases}
\end{align*}
\]

ii. We solve each one of the two associated equations.

a. The solution of the first associated equation is \(-786.33\). So \(-786.33\) is a boundary point of the solution subset of the double problem.

b. The solution of the second associated equation is \(+315.32\). So \(+315.32\) is a boundary point of the solution subset of the double problem.

iii. We check each boundary points to find if it is included or non-included in the solution subset of the double problem.

a. We check the boundary point \(-786.33\). Since

\[
\text{either one or both } \begin{cases} 
  x \neq -786.33 \\
  x \geq +315.32 
\end{cases} \quad x = -786.33
\]

the boundary point \(-786.33\) is non-included in the solution subset of the double problem.

b. We check the boundary point \(+315.32\). Since

\[
\text{either one or both } \begin{cases} 
  x \neq -786.33 \\
  x \geq +315.32 
\end{cases} \quad x = +315.32
\]

the boundary point \(+315.32\) is included in the solution subset of the double problem.

iv. The graph of the boundary of the solution subset of the double problem is therefore:

\[\text{Dollars} \quad -786.33 \quad +315.32\]

II. To locate the interior of the solution subset of the double problem, we use the PASCH PROCEDURE:

i. The boundary points split the data set in three regions which we label Section A, Section B and Section C as follows:
ii. We test Section A with, for instance, the point $-1000$. Since

\[
\text{EITHER ONE OR BOTH } \begin{cases} x \neq -786.33 \\ x \geq +315.32 \end{cases} \quad \text{is true}
\]

the point $-1000$ is a solution of the double problem in Dollars.

EITHER ONE OR BOTH \[
\begin{cases} x \neq -786.33 \\ x \geq +315.32 \end{cases}
\]

and Pasch's Theorem then tells us that all points in Section A are included in the solution subset of the double problem.

iii. We test Section B with, for instance, the point $0$. Since

\[
\text{EITHER ONE OR BOTH } \begin{cases} x \neq -786.33 \\ x \geq +315.32 \end{cases} \quad \text{is true}
\]

the point $0$ is a solution of the double problem in Dollars.

EITHER ONE OR BOTH \[
\begin{cases} x \neq -786.33 \\ x \geq +315.32 \end{cases}
\]

and Pasch's Theorem then tells us that all points in Section B are included in the solution subset of the double problem.

iv. We test Section C with, for instance, the point $+1000$. Since

\[
\text{EITHER ONE OR BOTH } \begin{cases} x \neq -786.33 \\ x \geq +315.32 \end{cases} \quad \text{is true}
\]

the point $+1000$ is a solution of the double problem in Dollars.

EITHER ONE OR BOTH \[
\begin{cases} x \neq -786.33 \\ x \geq +315.32 \end{cases}
\]

and Pasch's Theorem then tells us that all points in Section C are included in the solution subset of the double problem.

III. Altogether, the graph of the solution subset of the double problem in Dollars.
21. Question: Given the double basic problem in Dollars

Either one or both
\[
\begin{align*}
x & \neq -786.33 \\
x & \geq +315.32
\end{align*}
\]

is therefore:

```
-786.33
```

Dollars

What is the graph of its solution set?

Discussion:

I. To locate the boundary of the solution subset of the double problem, we must solve the associated problem.

   i. The associated problem in Dollars is:

   OR
   \[
   \begin{align*}
x & = -786.33 \\
x & = +315.32
\end{align*}
\]

   ii. We solve each one of the two associated equations.

   a. The solution of the first associated equation is \(-786.33\) so \(-786.33\) is a boundary point of the solution subset of the double problem.

   b. The solution of the second associated equation is \(+315.32\) so \(+315.32\) is a boundary point of the solution subset of the double problem.

   iii. We check each boundary points to find if it is included or non-included in the solution subset of the double problem.

   a. We check the boundary point \(-786.33\). Since

   \[
   \begin{align*}
   x & \leq -786.33 \\
   x & \geq +315.32
   \end{align*}
   \]

   is true

   the boundary point \(-786.33\) is included in the solution subset of the double problem.

   b. We check the boundary point \(+315.32\). Since
either one or both \( \begin{align*} x \leq & -786.33 \\ x \geq & +315.32 \end{align*} \) is true

the boundary point +315.32 is included in the solution subset of the double problem.

iv. The graph of the boundary of the solution subset of the double problem is therefore:

\[ 
\begin{array}{c}
-786.33 \\
\bullet \\
\bullet \\
+315.32 \\
\rightarrow \text{Dollars} \\
\end{array}
\]

II. To locate the interior of the solution subset of the double problem, we use the Pasch Procedure:

i. The boundary points split the data set in three regions which we label Section A, Section B and Section C as follows:

\[ 
\begin{array}{c}
\text{Section A} \\
-786.33 \\
\bullet \\
\bullet \\
\text{Section B} \\
+315.32 \\
\rightarrow \text{Dollars} \\
\end{array}
\]

ii. We test Section A with, for instance, the point -1000. Since

either one or both \( \begin{align*} x \leq & -786.33 \\ x \geq & +315.32 \end{align*} \) is true

the point -1000 is a solution of the double problem in Dollars

\[ 
\text{either one or both} \begin{align*} x \leq & -786.33 \\ x \geq & +315.32 \end{align*} \]

and Pasch’s Theorem then tells us that all points in Section A are included in the solution subset of the double problem.

iii. We test Section B with, for instance, the point 0. Since

either one or both \( \begin{align*} x \leq & -786.33 \\ x \geq & +315.32 \end{align*} \) is false

the point 0 is a non-solution of the double problem in Dollars

\[ 
\text{either one or both} \begin{align*} x \leq & -786.33 \\ x \geq & +315.32 \end{align*} \]
and Pasch’s Theorem then tells us that all points in Section B are non-included in the solution subset of the double problem.

iv. We test Section C with, for instance, the point +1000. Since

\[
\text{EITHER ONE OR BOTH} \begin{cases} 
    x \leq -786.33 \\
    x \geq +315.32 \\
\end{cases} \ x = +1000
\]

is true

the point +1000 is a solution of the double problem in Dollars

\[
\text{EITHER ONE OR BOTH} \begin{cases} 
    x \leq -786.33 \\
    x \geq +315.32 \\
\end{cases}
\]

and Pasch’s Theorem then tells us that all points in Section C are included in the solution subset of the double problem.

III. Altogether, the graph of the solution subset of the double problem in Dollars

\[
\text{EITHER ONE OR BOTH} \begin{cases} 
    x \leq -786.33 \\
    x \geq +315.32 \\
\end{cases}
\]

is therefore:

\[\text{Dollars} \]

\[\begin{array}{c}
-786.33 \\
\hline
+315.32
\end{array}\]

22. Question: Given the double basic problem in Dollars

\[
\text{EITHER ONE BUT NOT BOTH} \begin{cases} 
    x \leq +786.33 \\
    x \geq +315.32 \\
\end{cases}
\]

what is the graph of its solution set?

Discussion:

I. To locate the boundary of the solution subset of the double problem, we must solve the associated problem.

i. The associated problem in Dollars is:

\[
\text{OR} \begin{cases} 
    x = +786.33 \\
    x = +315.32 \\
\end{cases}
\]

ii. We solve each one of the two associated equations.
a. The solution of the first associated equation is +786.33. So +786.33 is a boundary point of the solution subset of the double problem.

b. The solution of the second associated equation is +315.32. So +315.32 is a boundary point of the solution subset of the double problem.

iii. We check each boundary points to find if it is included or non-included in the solution subset of the double problem.

a. We check the boundary pointt +786.33. Since

\[
\text{EITHER ONE BUT NOT BOTH } \begin{cases} x \leq +786.33 \\ x \geq +315.32 \end{cases} \quad x := +786.33
\]

the boundary point +786.33 is non-included in the solution subset of the double problem.

b. We check the boundary pointt +315.32. Since

\[
\text{EITHER ONE BUT NOT BOTH } \begin{cases} x \leq +786.33 \\ x \geq +315.32 \end{cases} \quad x := +315.32
\]

the boundary point +315.32 is non-included in the solution subset of the double problem.

iv. The graph of the boundary of the solution subset of the double problem is therefore:

\[\begin{array}{c}
\text{Dollars} \\
+315.32 \\
+786.33
\end{array}\]

II. To locate the interior of the solution subset of the double problem, we use the Pasch Procedure:

i. The boundary points split the data set in three regions which we label Section A, Section B and Section C as follows:

\[\begin{array}{c}
\text{Dollars} \\
+315.32 \\
+786.33
\end{array}\]

ii. We test Section A with, for instance, the point −1000. Since

\[
\text{EITHER ONE BUT NOT BOTH } \begin{cases} x \leq +786.33 \\ x \geq +315.32 \end{cases} \quad x := -1000
\]

is true
the point $-1000$ is a solution of the double problem in Dollars

$$\text{EITHER ONE BUT NOT BOTH } \begin{cases} x \leq +786.33 \\ x \geq +315.32 \end{cases}$$

and Pasch’s Theorem then tells us that all points in Section A are included in the solution subset of the double problem.

iii. We test Section B with, for instance, the point 400 (Here 0 is not in Section B). Since

$$\text{EITHER ONE BUT NOT BOTH } \begin{cases} x \leq +786.33 \\ x \geq +315.32 \end{cases} \bigg|_{x=400}$$

the point 400 is a non-solution of the double problem in Dollars

$$\text{EITHER ONE BUT NOT BOTH } \begin{cases} x \leq +786.33 \\ x \geq +315.32 \end{cases}$$

and Pasch’s Theorem then tells us that all points in Section B are non-included in the solution subset of the double problem.

iv. We test Section C with, for instance, the point $+1000$. Since

$$\text{EITHER ONE BUT NOT BOTH } \begin{cases} x \leq +786.33 \\ x \geq +315.32 \end{cases} \bigg|_{x=+1000}$$

the point $+1000$ is a solution of the double problem in Dollars

$$\text{EITHER ONE BUT NOT BOTH } \begin{cases} x \leq +786.33 \\ x \geq +315.32 \end{cases}$$

and Pasch’s Theorem then tells us that all points in Section C are included in the solution subset of the double problem.

III. Altogether, the graph of the solution subset of the double problem in Dollars

$$\text{EITHER ONE BUT NOT BOTH } \begin{cases} x \leq +786.33 \\ x \geq +315.32 \end{cases}$$

is therefore:

23. Question: Given the double basic problem in Dollars
either one but not both \( \begin{cases} x > -786.33 \\ x < +315.32 \end{cases} \)

what is the graph of its solution set?

**Discussion:**

I. To locate the boundary of the solution subset of the double problem, we must solve the associated problem.

i. The associated problem in Dollars is:

\[
\text{OR} \begin{cases} x = -786.33 \\ x = +315.32 \end{cases}
\]

ii. We solve each one of the two associated equations.

a. The solution of the first associated equation is \(-786.33\). So \(-786.33\) is a boundary point of the solution subset of the double problem.

b. The solution of the second associated equation is \(+315.32\). So \(+315.32\) is a boundary point of the solution subset of the double problem.

iii. We check each boundary points to find if it is included or non-included in the solution subset of the double problem.

a. We check the boundary point \(-786.33\). Since

\[
\text{either one but not both} \begin{cases} x > -786.33 \\ x < +315.32 \end{cases} \quad \text{is true} \\
\text{with } x = -786.33
\]

the boundary point \(-786.33\) is included in the solution subset of the double problem.

b. We check the boundary point \(+315.32\). Since

\[
\text{either one but not both} \begin{cases} x > -786.33 \\ x < +315.32 \end{cases} \quad \text{is true} \\
\text{with } x = +315.32
\]

the boundary point \(+315.32\) is included in the solution subset of the double problem.

iv. The graph of the boundary of the solution subset of the double problem is therefore:
II. To locate the interior of the solution subset of the double problem, we use the PASCH PROCEDURE:

i. The boundary points split the data set in three regions which we label Section A, Section B and Section C as follows:

```
<table>
<thead>
<tr>
<th>Section A</th>
<th>Section B</th>
<th>Section C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-786.33$</td>
<td>$+315.32$</td>
<td>$+315.32$</td>
</tr>
</tbody>
</table>
```

ii. We test Section A with, for instance, the point $-1000$. Since

\[
\begin{align*}
\text{either one but not both} & \quad \left\{ x > -786.33, x < +315.32 \right\} \\
& \quad \text{is TRUE for } x = -1000
\end{align*}
\]

the point $-1000$ is a solution of the double problem in Dollars

\[
\begin{align*}
\text{either one but not both} & \quad \left\{ x > -786.33, x < +315.32 \right\}
\end{align*}
\]

and Pasch’s Theorem then tells us that all points in Section A are included in the solution subset of the double problem.

iii. We test Section B with, for instance, the point $0$. Since

\[
\begin{align*}
\text{either one but not both} & \quad \left\{ x > -786.33, x < +315.32 \right\} \\
& \quad \text{is FALSE for } x = 0
\end{align*}
\]

the point $0$ is a non-solution of the double problem in Dollars

\[
\begin{align*}
\text{either one but not both} & \quad \left\{ x > -786.33, x < +315.32 \right\}
\end{align*}
\]

and Pasch’s Theorem then tells us that all points in Section B are non-included in the solution subset of the double problem.

iv. We test Section C with, for instance, the point $+1000$. Since

\[
\begin{align*}
\text{either one but not both} & \quad \left\{ x > -786.33, x < +315.32 \right\} \\
& \quad \text{is TRUE for } x = +1000
\end{align*}
\]
the point +1000 is a solution of the double problem in Dollars

\[
\begin{align*}
\text{EITHER ONE BUT NOT BOTH} & \quad \begin{cases} 
    x > -786.33 \\
    x < +315.32 
\end{cases}
\end{align*}
\]

and Pasch’s Theorem then tells us that all points in Section C are included in the solution subset of the double problem.

III. Altogether, the graph of the solution subset of the double problem in Dollars

\[
\begin{align*}
\text{EITHER ONE BUT NOT BOTH} & \quad \begin{cases} 
    x > -786.33 \\
    x < +315.32 
\end{cases}
\end{align*}
\]

is therefore:

---

24. Question: Given the double basic problem in Dollars

\[
\begin{align*}
\text{EITHER ONE OR BOTH} & \quad \begin{cases} 
    x \geq +925.04 \\
    x < -394.96 
\end{cases}
\end{align*}
\]

what is the graph of its solution set?

Discussion:

I. To locate the boundary of the solution subset of the double problem, we must solve the associated problem.

i. The associated problem in Dollars is:

\[
\begin{align*}
\text{OR} & \quad \begin{cases} 
    x = +925.04 \\
    x = -394.96 
\end{cases}
\end{align*}
\]

ii. We solve each one of the two associated equations.

a. The solution of the first associated equation is +925.04. So +925.04 is a boundary point of the solution subset of the double problem.

b. The solution of the second associated equation is −394.96. So −394.96 is a boundary point of the solution subset of the double problem.

iii. We check each boundary points to find if it is included or non-included in the solution subset of the double problem.
a. We check the boundary point $+925.04$. Since 

$$\begin{align*}
\text{EITHER ONE OR BOTH} & \quad \begin{cases} 
  x \geq +925.04 \\
  x < -394.96 
\end{cases} \\
& \quad x = +925.04
\end{align*}$$

the boundary point $+925.04$ is *included* in the solution subset of the double problem.

b. We check the boundary point $-394.96$. Since 

$$\begin{align*}
\text{EITHER ONE OR BOTH} & \quad \begin{cases} 
  x \geq +925.04 \\
  x < -394.96 
\end{cases} \\
& \quad x = -394.96
\end{align*}$$

the boundary point $-394.96$ is *non-included* in the solution subset of the double problem.

iv. The graph of the *boundary* of the solution subset of the double problem is therefore:

II. To locate the *interior* of the solution subset of the double problem, we use the *Pasch Procedure*:

i. The boundary points split the data set in three regions which we label Section A, Section B and Section C as follows:

```
Section A       Section B       Section C
```

```
-394.96        +925.04        Dollars
```

ii. We test Section A with, for instance, the point $-1000$. Since 

$$\begin{align*}
\text{EITHER ONE OR BOTH} & \quad \begin{cases} 
  x \geq +925.04 \\
  x < -394.96 
\end{cases} \\
& \quad x = -1000
\end{align*}$$

the point $-1000$ is a *solution* of the double problem in *Dollars*

$$\begin{align*}
\text{EITHER ONE OR BOTH} & \quad \begin{cases} 
  x \geq +925.04 \\
  x < -394.96 
\end{cases}
\end{align*}$$

and **Pasch’s Theorem** then tells us that all points in Section A are *included* in the solution subset of the double problem.
iii. We test Section B with, for instance, the point 0. Since

\[
\text{EITHER ONE OR BOTH} \left\{ \begin{array}{l}
x \geq +925.04 \\
x < -394.96
\end{array} \right. \bigg|_{x=0}
\]

is FALSE

the point 0 is a non-solution of the double problem in Dollars

\[
\text{EITHER ONE OR BOTH} \left\{ \begin{array}{l}
x \geq +925.04 \\
x < -394.96
\end{array} \right.
\]

and Pasch’s Theorem then tells us that all points in Section B are non-included in the solution subset of the double problem.

iv. We test Section C with, for instance, the point +1000. Since

\[
\text{EITHER ONE OR BOTH} \left\{ \begin{array}{l}
x \geq +925.04 \\
x < -394.96
\end{array} \right. \bigg|_{x=+1000}
\]

is TRUE

the point +1000 is a solution of the double problem in Dollars

\[
\text{EITHER ONE OR BOTH} \left\{ \begin{array}{l}
x \geq +925.04 \\
x < -394.96
\end{array} \right.
\]

and Pasch’s Theorem then tells us that all points in Section C are included in the solution subset of the double problem.

III. Altogether, the graph of the solution subset of the double problem in Dollars

\[
\text{EITHER ONE OR BOTH} \left\{ \begin{array}{l}
x \geq +925.04 \\
x < -394.96
\end{array} \right.
\]

is therefore:

25. **Question:** Given the double affine problem in Dollars

\[
\text{BOTH} \left\{ \begin{array}{l}
+2x + 4 \geq -2 \\
+3x - 4 < +8
\end{array} \right.
\]

what is the graph of its solution subset?

**Discussion:**
I. To locate the boundary of the solution subset, we must solve the associated problem.

i. The associated problem in Dollars is:

\[
\begin{aligned}
&+2x + 4 = -2 \\
&+3x - 4 = +8
\end{aligned}
\]

ii. We use the Reduction Approach to solve each one of the two associated equations.

a. The first associated equation is an affine equation which we reduce to a dilation equation and then to a basic equation:

\[
\begin{aligned}
&+2x + 4 = -2 \\
&+2x + 4 \oplus -4 = -2 \oplus -4 \\
&+2x = -6 \\
&+2x \div +2 = -6 \div +2 \\
&x = -3
\end{aligned}
\]

which, by the Fairness Theorem, is equivalent to the original equation. So, −3 is a boundary point of the original double affine problem.

b. The second associated equation is an affine equation which we reduce to a dilation equation and then to a basic equation:

\[
\begin{aligned}
&+3x - 4 = +8 \\
&+3x - 4 \oplus +4 = +8 \oplus +4 \\
&+3x = +12 \\
&+3x \div +3 = +12 \div +3 \\
&x = +4
\end{aligned}
\]

which, by the Fairness Theorem, is equivalent to the original equation. So, +4 is a boundary point of the original double affine problem.

iii. We check if each of the boundary points are included or non-included in the solution subset.

a. Since the first inequation in the problem involves \( \leq \), the boundary point −3 is included in the solution subset.

b. Since the second inequation in the problem involves \( > \), the boundary point +4 is non-included in the solution subset.
iv. The graph of the boundary of the solution subset of the problem is therefore:

\[\begin{array}{c}
\text{\$} \quad \text{\$} \\
\end{array}\]

II. To locate the interior of the solution subset, we use the General Procedure:

i. The boundary points split the data set in three regions which we label Section A, Section B and Section C as follows:

\[\begin{array}{c}
\text{Section A} \quad \text{Section B} \quad \text{Section C} \\
\end{array}\]

ii. We test Section A with, for instance, \(-100\) and, since

\[
\text{BOTH}\begin{cases}
+2x + 4 \geq -2 \\
+3x - 4 < +2
\end{cases}
\text{ is FALSE}
\]

we get that \(-100\) is a non-solution of the double affine problem in Dollars

\[
\text{BOTH}\begin{cases}
+2x + 4 \geq -2 \\
+3x - 4 < +2
\end{cases}
\]

and Pasch’s Theorem then tells us that all number-phrases in Section A are non-included in the solution subset.

iii. We test Section B with, for instance, 0 and, since

\[
\text{BOTH}\begin{cases}
+2x + 4 \geq -2 \\
+3x - 4 < +2
\end{cases}
\text{ is TRUE}
\]

we get that 0 is a solution of the double affine problem in Dollars

\[
\text{BOTH}\begin{cases}
+2x + 4 \geq -2 \\
+3x - 4 < +2
\end{cases}
\]

and Pasch’s Theorem then tells us that all number-phrases in Section B are included in the solution subset.

iv. We test Section C with, for instance, +100 and, since

\[
\text{BOTH}\begin{cases}
+2x + 4 \geq -2 \\
+3x - 4 < +2
\end{cases}
\text{ is FALSE}
\]
we get that +100 is a non-solution of the double affine problem in Dollars

\[
\begin{align*}
\text{both} & \quad +2x + 4 \geq -2 \\
& \quad +3x - 4 < +2
\end{align*}
\]

and Pasch’s Theorem then tells us that all number-phrases in Section C are non-included in the solution subset.

**III.** Altogether, the graph of the solution subset of the problem in Dollars

\[
\begin{align*}
\text{both} & \quad +2x + 4 \geq -2 \\
& \quad +3x - 4 < +2
\end{align*}
\]

is therefore:
1. **Question:** Identify the monomial specifying-phrase in **dollars**

$$3 \times 5^4$$

**Discussion:** We go through the following steps:
- We *read* $3 \times 5^4$ as *3 multiplied* by 4 copies of 5
- We *write:* $3 \times 5^4 = 3 \times 5 \times 5 \times 5 \times 5$
- We *compute:* $= 1875$

2. **Question:** Identify the monomial specifying-phrase in **dollars**

$$256 \times 2^{-5}$$

**Discussion:** We go through the following steps:
- We *read* $256 \times 2^{-5}$ as *256 divided* by 5 copies of 2
- We *write:* $256 \times 2^{-5} = \frac{256}{2 \times 2 \times 2 \times 2 \times 2}$
- We *compute:* $= \frac{256}{32} = 8$

3. **Question:** Identify the monomial specifying-phrase in **dollars**

$$-162 \times (-3)^{-2}$$

**Discussion:**
- We *read* $-162 \times (-3)^{-2}$ as *-162 divided* by 2 copies of -3
- We *write:* $-162 \times (-3)^{-2} = \frac{-162}{(-3) \times (-3)}$
- We *compute:* $= -18$

4. **Question:** Identify the monomial specifying-phrase in **dollars**

$$[10 \times 2^{-5}] \times [6 \times 2^{+3}]$$

**Discussion:** In order to *multiply* monomial specifying-phrase with a common basis,
i. We get the coefficient of the resulting monomial specifying-phrase by multiplying the coefficients of the given monomial specifying-phrases:

\[10 \times 6\]

ii. We get the base of the resulting monomial specifying-phrase by taking the base common to the given monomial specifying-phrases:

\[2\]

iii. We get the signed exponent of the resulting monomial specifying-phrase by “oplussing” the signed exponents of the given monomial specifying-phrases:

\[-5 \ominus +3\]

So, we write and compute:

\[
\left[10 \times 2^{-5}\right] \times \left[6 \times 2^{+3}\right] = [10 \times 6] \times 2^{-5 \ominus +3}
\]

\[= 60 \times 2^{-2}\]

\[= \frac{60}{2 \times 2}\]

\[= 15\]

5. Question: Identify the specifying-phrase in Dollars

\[
[512 \times 3^{-5}] \div [4 \times 3^{+3}]\]

Discussion: In order to divide monomial specifying-phrase with a common basis,

i. We get the coefficient of the resulting monomial specifying-phrase by dividing the coefficients of the given monomial specifying-phrases:

\[512 \div 4\]

ii. We get the base of the resulting monomial specifying-phrase by taking the base common to the given monomial specifying-phrases:

\[3\]

iii. We get the signed exponent of the resulting monomial specifying-phrase by “ominussing” the signed exponents of the given monomial specifying-phrases:

\[-5 \ominus +3\]
So, we write and compute:

\[
\left[ 512 \times 3^{-5} \right] \div \left[ 4 \times 3^{+3} \right] = \left[ 512 \div 4 \right] \times 2^{-5 \oplus 3} = 128 \times 2^{-8} = \frac{128}{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2} = 0.5
\]

6. **Question:** Identify \([-12x^{+5}] \otimes [-4x^{-3}]\)

**Discussion:** In order to multiply monomial specifying-phrase with a common basis,

i. We get the coefficient of the resulting monomial specifying-phrase by multiplying the coefficients of the given monomial specifying-phrases:

\[-12 \otimes -4\]

ii. We get the base of the resulting monomial specifying-phrase by taking the base common to the given monomial specifying-phrases:

\[x\]

iii. We get the signed exponent of the resulting monomial specifying-phrase by “oplussing” the signed exponents of the given monomial specifying-phrases:

\[+5 \oplus -3\]

So, we write and compute:

\[
\left[-12x^{+5}\right] \otimes \left[-4x^{-3}\right] = \left[-12 \otimes -4\right] x^{+5 \oplus -3} = -48x^{+2}
\]

(Since \(x\) is unspecified we cannot go any further.)

7. **Question:** Identify \([+12x^{-5}] \otimes [+4x^{+3}]\)

**Discussion:** In order to multiply monomial specifying-phrase with a common basis,

i. We get the coefficient of the resulting monomial specifying-phrase by multiplying the coefficients of the given monomial specifying-phrases:
\[ +12 \otimes +4 \]

ii. We get the base of the resulting monomial specifying-phrase by taking the base common to the given monomial specifying-phrases:

\[ x \]

iii. We get the signed exponent of the resulting monomial specifying-phrase by “oplussing” the signed exponents of the given monomial specifying-phrases:

\[ -5 \oplus +3 \]

So, we write and compute:

\[
\begin{align*}
[+12x^{-5}] \otimes [+4x^{+3}] &= [+12 \otimes +4] x^{-5} \oplus +3 \\
&= +48x^{-2}
\end{align*}
\]

(Since \( x \) is unspecified we cannot go any further.)

8. Question: Identify \([+12x^{-5}] \otimes [-4x^{-3}]\)

Discussion: In order to multiply monomial specifying-phrase with a common basis,

i. We get the coefficient of the resulting monomial specifying-phrase by multiplying the coefficients of the given monomial specifying-phrases:

\[ +12 \otimes -4 \]

ii. We get the base of the resulting monomial specifying-phrase by taking the base common to the given monomial specifying-phrases:

\[ x \]

iii. We get the signed exponent of the resulting monomial specifying-phrase by “oplussing” the signed exponents of the given monomial specifying-phrases:

\[ -5 \oplus -3 \]

So, we write and compute:

\[
\begin{align*}
[+12x^{-5}] \otimes [-4x^{+3}] &= [+12 \otimes -4] x^{-5} \oplus +3 \\
&= -48x^{-2}
\end{align*}
\]

(Since \( x \) is unspecified we cannot go any further.)
9. **Question:** Identify \([+12x^5] \oplus [-4x^3]\)

**Discussion:** In order to *divide* monomial specifying-phrase with a common basis,

i. We get the *coefficient* of the resulting monomial specifying-phrase by *dividing* the coefficients of the given monomial specifying-phrases:

\[+12 \oplus -4\]

ii. We get the *base* of the resulting monomial specifying-phrase by taking the base common to the given monomial specifying-phrases:

\[x\]

iii. We get the signed exponent of the resulting monomial specifying-phrase by “ominussing” the signed exponents of the given monomial specifying-phrases:

\[+5 \ominus -3\]

So, we write and compute:

\[
\begin{array}{l}
[+12x^{-5}] \oplus [-4x^{+3}] = [+12 \oplus -4] x^{+5 \ominus -3} \\
= [+12 \oplus -4] x^{+5 \oplus +3} \\
= -3x^{+8}
\end{array}
\]

(Since \(x\) is unspecified we cannot go any further.)

10. **Question:** Identify \([-12x^{-5}] \oplus [-4x^{+3}]\)

**Discussion:** In order to *divide* monomial specifying-phrase with a common basis,

i. We get the *coefficient* of the resulting monomial specifying-phrase by *dividing* the coefficients of the given monomial specifying-phrases:

\[-12 \oplus -4\]

ii. We get the *base* of the resulting monomial specifying-phrase by taking the base common to the given monomial specifying-phrases:

\[x\]

iii. We get the signed exponent of the resulting monomial specifying-phrase by “ominussing” the signed exponents of the given monomial specifying-phrases:

\[-5 \ominus +3\]
So, we write and compute:

\[
\begin{align*}
[-12x^{-5}] & \otimes [-4x^3] = [-12 \otimes -4]x^{-5} \oplus^3 \\
& = [-12 \otimes -4]x^{-5} \oplus -3 \\
& = +3x^{-8}
\end{align*}
\]

(Since \(x\) is unspecified we cannot go any further.)

11. **Question:** Identify \([-12x^{-5}] \otimes [+4x^{-3}]\)

**Discussion:** In order to *divide* monomial specifying-phrase with a common basis,

i. We get the *coefficient* of the resulting monomial specifying-phrase by *dividing* the coefficients of the given monomial specifying-phrases:

\[-12 \otimes 4\]

ii. We get the *base* of the resulting monomial specifying-phrase by taking the base common to the given monomial specifying-phrases:

\(x\)

iii. We get the signed exponent of the resulting monomial specifying-phrase by “ominussing” the signed exponents of the given monomial specifying-phrases:

\[-5 \oplus -3\]

So, we write and compute:

\[
\begin{align*}
[-12x^{-5}] & \otimes [+4x^3] = [-12 \otimes +4]x^{-5} \oplus -3 \\
& = [-12 \otimes +4]x^{-5} \oplus +3 \\
& = -3 \times x^{-2}
\end{align*}
\]

(Since \(x\) is unspecified we cannot go any further.)

12. **Question:** Identify \([-3x^2 + 6x - 2 + x^{-1} - 2x^{-2}] \oplus [4x^2 - x + 3 + 2x^{-1} - 3x^{-2}]\)

**Discussion:** We write-in what “goes without saying” and line up vertically the monomials that can be *added* (because they have the same exponent):

\[-3x^2 + 6x - 2 + 1x^{-1} - 2x^{-2}\]

\oplus

\[+4x^2 - x + 3 + 2x^{-1} - 3x^{-2}\]
13. Question: Identify \([ -3x^2 + 1 - 2x^{-1} ] \oplus [ 4x^3 - x^2 + 3x + 2 - 3x^{-1} ]\)

Discussion: We write-in what “goes without saying” and line up vertically the monomials that can be added (because they have the same exponent):

\[
\begin{align*}
-3x^2 + 0x + 1 - 2x^{-1} \\
+ 4x^3 - x^2 + 3x + 2 - 3x^{-1}
\end{align*}
\]

\[+4x^3 - 4x^2 + 3x + 4 - 5x^{-1}\]

14. Question: Identify \(-3x^2 + x^{-1} - 2x^{-2} + [4x^3 - x + 3 + 2x^{-1} - 3x^{-2}]\)

Discussion: We write-in what “goes without saying” and line up vertically the monomials that can be added (because they have the same exponent):

\[
\begin{align*}
-3x^2 + 0x + 0 + 1x^{-1} - 2x^{-2} \\
+ 4x^3 + 0x^2 - x + 3 + 2x^{-1} - 3x^{-2}
\end{align*}
\]

\[+4x^3 - 3x^2 - x + 3 + 3x^{-1} - 5x^{-2}\]

15. Question: \([ -3x + 6 - 2x^{-1} + x^{-2} - 2x^{-3} ] \oplus [ 4x - 1 + 3x^{-1} + 2x^{-2} - 3x^{-3} ]\)

Discussion:

- In order to subtract a polynomial, we add the opposite of this polynomial.
- We write-in what “goes without saying” and line up vertically the monomials that can be added (because they have the same exponent):

\[
\begin{align*}
-3x + 6 - 2x^{-1} + 1x^{-2} - 2x^{-3} \\
-4x + 1 - 3x^{-1} - 2x^{-2} + 3x^{-3}
\end{align*}
\]

\[-7x + 7 - 5x^{-1} - x^{-2} + 1x^{-3}\]
16. **Question:** Identify \([-3 + x^{-2} - 2x^{-3}] □ [4x - 1 + 3x^{-1} + 2x^{-2} - 3x^{-3} + 5x^{-4}]\)

**Discussion:**
- In order to *subtract* a polynomial, we *add the opposite of* this polynomial.
- We write-in what “goes without saying” and line up vertically the monomials that can be *added* (because they have the same exponent):

\[
\begin{align*}
-3 + 0x^{-1} + x^{-2} - 2x^{-3} &
\oplus
-4x + 1 - 3x^{-1} - 2x^{-2} + 3x^{-3} - 5x^{-4} \\
\hline
-4x - 2 - 3x^{-1} - x^{-2} + x^{-3} - 5x^{-4}
\end{align*}
\]

17. **Question:** Identify \(-3x + x^{-1} - [4x^3 - x^2 + 3x + 2 - 3x^{-1}]\)

**Discussion:** The \(-\) in front of the \([\text{ is really a } □\text{ and there is a } +\text{ that goes “without saying” in front of the } 4x^2.\)
- In order to *subtract* a polynomial, we *add the opposite of* this polynomial.
- We write-in what “goes without saying” and line up vertically the monomials that can be *added* (because they have the same exponent):

\[
\begin{align*}
-3x + 0 + 1x^{-1} &
\oplus
-4x^3 + 1x^2 - 3x - 2 + 3x^{-1} \\
\hline
-4x^3 + 1x^2 - 6x - 2 + 4x^{-1}
\end{align*}
\]

18. **Question:** Identify \([5x - 6 + 7x^{-1}] □ [-3x + 5]\)

**Discussion:** Since polynomials are combinations of monomials, multiplication of polynomials is essentially multiplication of monomials. The additional step is to reduce the result by adding “like” monomials which is the reason for the layout we use.

\[
\begin{align*}
5x - 6 + 7x^{-1} &
\otimes
-3x + 5 \\
\hline
-15x^2 + 18x - 21
\end{align*}
\]
+25x − 30 + 35x^{-1}

\[ -15x^2 + 43x - 51 + 35x^{-1} \]

19. **Question:** Identify \([-5x^5 + 2x^3 - 6x^2] \otimes [1 - 4x^{-2}]\)

**Discussion:** Since polynomials are combinations of monomials, multiplication of polynomials is essentially multiplication of monomials. The additional step is to reduce the result by adding “like” monomials which is the reason for the layout we use.

We write-in what “goes without saying”.

\[
-5x^5 + 0x^4 + 2x^3 - 6x^2 \\
\otimes \\
+1 + 0x^{-1} - 4x^{-2}
\]

\[
-5x^5 + 0x^4 + 2x^3 - 6x^2 \\
+20x^2 - 8x + 24
\]

\[
-5x^5 + 0x^4 + 2x^3 + 14x^2 - 8x + 24
\]

20. **Question:** Identify \((-8 + h)^2\)

**Discussion:** We can look at the question from two points of view (but we certainly should not multiply two copies of \(-8 + h\)).

- In a (pseudo) real-world situation, we are looking at the area of a square the “length” of whose side used to be \(-8\) but which has now been increased by \(h\) to \(-8 + h\). In other words, we are looking at the following picture:

![Diagram of a square with side lengths](image)
We then get the area of the new square by adding to
The original \(-8\) by \(-8\) square = \((-8)^2\):
  Two \(-8\) by \(h\) strips = \(2 \cdot (-8)h\)
  The little \(h\) by \(h\) square = \(h^2\)

- On paper, we get the binomial expansion as follows:
  i. We construct the successive powers:
     \((-8)^2h^0\)
     \((-8)^1h^1\)
     \((-8)^0h^2\)
  which we write horizontally:
    \((-8)^2h^0\) \((-8)^1h^1\) \((-8)^0h^2\)
  ii. We write the successive coefficients with the aid of the Pascal Triangle:
    \[\begin{array}{cccc}
    n := 0 & 1 \\
    n := 1 & 1 & 1 \\
    n := 2 & 1 & 2 & 1
    \end{array}\]
  iii. We assemble the powers and the coefficients:
    \[1 \cdot (-8)^2h^0 + 2 \cdot (-8)^1h^1 + 1 \cdot (-8)^0h^2\]
    Either way, after computations, we end up with the binomial expansion
    \[+64 - 16h + h^2\]

21. Question: Identify \((-7 + h)^3\)

Discussion: We can look at the question from two points of view (but we certainly should not multiply three copies of \(-7 + h\)).

- In a (pseudo) real-world situation, we are looking at the volume of a cube the “length” of whose side used to be \(-7\) but which has now been increased by \(h\) to \(-7 + h\). In other words, we are looking at the following picture:
We then get the volume of the new cube by adding to
The original $-7 \times -7 \times -7$ cube $= (-7)^3$:

- Three $-7 \times -7 \times h$-thick slabs $= 3 \cdot (-7)^2 h$
- Three $-7$-long $h$ by $h$ rods $= 3 \cdot (-7)h^2$
- The little $h$ by $h$ by $h$ cube $= h^3$

- On paper, we get the binomial expansion as follows:
  
  i. We construct the successive powers:
      
      \[
      (-7)^3 h^0 \\
      (-7)^2 h^1 \\
      (-7)^1 h^2 \\
      (-7)^0 h^3
      \]

  which we write horizontally:

      \[
      (-7)^3 h^0 \quad (-7)^2 h^1 \quad (-7)^1 h^2 \quad (-7)^0 h^3
      \]

  ii. We write the successive coefficients with the aid of the Pascal Triangle:

      \[
      \begin{array}{cccc}
      n & = & 0 & 1 \\
      n & = & 1 & 1 & 1 \\
      n & = & 2 & 1 & 2 & 1 \\
      n & = & 3 & 1 & 3 & 3 & 1
      \end{array}
      \]

  iii. We assemble the powers and the coefficients:

      \[
      1 \cdot (-7)^3 h^0 \quad + \quad 3 \cdot (-7)^2 h^1 \quad + \quad 3 \cdot (-7)^1 h^2 \quad + \quad 1 \cdot (-7)^0 h^3
      \]

  Either way, after computations, we end up with the binomial expansion

      \[-343 + 147h - 21h^2 + h^3\]

22. Question: Approximate $\frac{6x^3 - x^2 + 13x - 6}{3x - 2}$ to $x^1$.

Discussion: We divide $3x - 2$ into $6x^3 - x^2 + 13x - 6$:

\[
   \begin{array}{c|cccc}
   +2x^2 & +x \\
   +6x & -x^2 & +13x & -6 \\
   -6x & +4x^2 & \\
   +3x^2 & +13x & \\
\end{array}
\]

We write the approximation:

\[
\frac{6x^3 - x^2 + 13x - 6}{3x - 2} = +2x^2 + x + (...)\]
23. Question: Approximate $\frac{9x^3 - 13x + 6}{-3x^2 - x + 2}$ to $x^{-1}$

Discussion: We divide $-3x^2 - x + 2$ into $9x^3 - 13x + 6$:

\[
\begin{array}{cccc}
-3x & 1 & +2x^{-1} \\
9x^3 & +0x^2 & -13x & +6 \\
-9x^3 & -3x^2 & +6x \\
\hline
-3x^2 & -7x & +6 \\
+3x^2 & +x & -2 \\
\hline
-6x & +4 \\
\end{array}
\]

We write the approximation

\[
\frac{9x^3 - 13x + 6}{-3x^2 - x + 2} = -3x + 1 + 2x^{-1} + (\ldots)
\]

24. Question: Approximate $\frac{8 + 2h - 26h^2 + 10h^3}{4 - 3h}$ to $h^2$

Discussion: We divide $4 - 3h$ into $8 + 2h - 26h^2 + 10h^3$

\[
\begin{array}{cccc}
+2 & 2h & -5h^2 \\
+8 & +2h & -26h^2 & +10h^3 \\
-8 & +6h \\
\hline
+8h & -26h^2 & +10h^3 \\
-8h & +6h^2 \\
\hline
-20h^2 & +10h^3 \\
\end{array}
\]

We write the approximation:

\[
\frac{8 + 2h - 26h^2 + 10h^3}{4 - 3h} = 2 + 2h - 5h^2 + (\ldots)
\]

25. Question: Approximate $\frac{16 - 5h^2 + 7h^3}{4 - 3h + h^2}$ to $h^2$

Discussion: We divide $4 - 3h + h^2$ into $16 - 5h^2 + 7h^3$

\[
\begin{array}{cccc}
+4 & 2h & -5h^2 \\
+16 & +0h & -5h^2 & +7h^3 \\
-16 & +12h & -4h^2 \\
\hline
+8h & -26h^2 & +10h^3 \\
-8h & +6h^2 \\
\hline
-20h^2 & +10h^3 \\
\end{array}
\]
We write the approximation:

\[
\frac{8 + 2h - 26h^2 + 10h^3}{4 - 3h} = +4 + 2h - 5h^2 + (...) 
\]