Chapter 2

Comparisons:
Equalities and Inequalities

We investigate the first of the three fundamental processes involving two collections. We will introduce the procedure in the case of basic collections using basic counting number-phrases.

2.1 Counting From A Counting Number-Phrase To Another

Before we can develop the procedures for these three fundamental processes, we must make the concept of counting more flexible by allowing a count

- to start with any digit which we will call the start-digit. (So, the start-digit doesn’t have anymore to be 1 as it always did in Chapter 1.)
- to end with any digit which we will call the end-digit. (So, the end-digit may be “before” the start digit as well as “after” the start digit.)

Specifically, when we count from the start-digit to the end-digit:

i. We start (just) after the start-digit

ii. We stop (just) after the end-digit.

However, given a start-digit and an end-digit, we may have to count in either one of two possible directions:

• We may have to count-up, that is we may have to use the succession $1, 2, 3, 4, 5, 6, 7, 8, 9$.

which we read along the arrow, that is from left to right.

EXAMPLE 1. To count from the start-digit 3 to the end-digit 7:
i. We must count up, that is we must use the succession
\[1, 2, 3, 4, 5, 6, 7, 8, 9,\]

ii. We start counting up in the succession after the start-digit 3, so that 4 is the first digit we say,
\[4, \ldots\]

iii. We stop counting up in the succession after the end-digit 7 so that 7 is the last digit we say
\[\ldots 7\]

Altogether, the count from the start-digit 3 to the end-digit 7 is
\[4, 5, 6, 7\]

• We may have to count-down, that is we may have to use the precession
\[1, 2, 3, 4, 5, 6, 7, 8, 9\]
which we read along the arrow, that is from right to left.

NOTE. If we prefer to read from left to right, we may also write the precession as
\[9, 8, 7, 6, 5, 4, 3, 2, 1\]
which we read along the arrow, that is from left to right.

**Example 2.** To count from the start-digit 6 to the end-digit 2:

i. We must count down, that is we must use the precession
\[9, 8, 7, 6, 5, 4, 3, 2, 1\]

ii. We start counting down in the precession after the start-digit 6 so that 5 is the first digit we say
\[5, \ldots\]

iii. We stop counting down in the precession after the end-digit 2 so that 2 is the last digit we say.
\[\ldots 2\]

Altogether, the count from the start-digit 6 to the end-digit 2 is
\[5, 4, 3, 2\]

**NOTE.** Memorizing the precession \[9, 8, 7, 6, 5, 4, 3, 2, 1\] just like we memo-
rized the succession \[1, 2, 3, 4, 5, 6, 7, 8, 9\] makes life a lot easier.
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Finally, the **length of a count** from a start-digit to an end-digit is how many digits we say regardless of the direction, that is whether up in the succession or down in the precession.

**Example 3.** When we count from the start-digit 3 to the end-digit 7, the length of the count is 4.

**Example 4.** When we count from the start-digit 6 to the end-digit 2, the length of the count is 4.

What that does, as in Chapter 1, is again to separate *quality*—represented by the *direction* of the count, up or down, from *quantity*—represented by the *length* of the count, how many digits we count.

**Note.** As already mentioned, we will only use *basic* counting, whether up or down, but *extended* counting would work essentially the same way.

## 2.2 Comparing Collections

Given two collections, the first thing we usually want to do is to **compare** the first collection to the second collection but an immediate issue is whether the kinds of items in the two collections are the *same* or *different*.

- When the two given collections involve *different* kinds of items, they don’t they cannot be compared.

**Example 5.** If Jane’s collection is and Nell’s collection is , we don’t really want to compare them because that would mean that we are really looking at the items as and , that is that we would be ignoring some of the details in the pictures.

- When the two given collections involve the *same* kind of items, the real-world *process* we will use to compare the two collections will be to **match one-to-one** each item of the first collection with an item of the second collection and to look in which of the two collections the *leftover* items are in.

When the two given collections involve the *same* kind of items, there are *six* several different **relationships** that can **hold** from the first collection to the second collection.

1. Up front, we have two very **simple** relationships:
• When there are no leftover objects, we will say that the first collection is-the-same-in-size-as the second collection.

**Example 6.** To compare in the real-world Jack’s collection with Jill’s collection, we match Jack’s collection one-to-one with Jill’s collection:

Since there is no leftover item in either collection, the relationship between Jack’s collection and Jill’s collection is that:

Jack’s collection is-the-same-in-size-as Jill’s collection

• When there are leftover objects, regardless of where they are, we will say that the first collection is-different-in-size-from the second collection.

**Example 7.** To compare Jack’s collection with Jill’s collection in the real-world, we match Jack’s collection one-to-one with Jill’s collection:

Since there are leftover items in one of the two collections, the relationship between Jack’s collection and Jill’s collection is that:

Jack’s collection is-different-in-size-from Jill’s collection

**Example 8.** To compare in the real-world Jack’s collection with Jill’s collection, we match Jack’s collection one-to-one with Jill’s collection:

Since there are leftover items in one of the two collections, the relationship between Jack’s collection and Jill’s collection is that:
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Jack’s collection is different-in-size-from Jill’s collection

2. When two collections are different-in-size, then there are two possible **strict** relationships depending on which of the two collections the leftover item, if any, are in:

- When the leftover items are in the second collection, we will say that the first collection is smaller-in-size-than the second collection.

**Example 9.** To compare Jack’s with Jill’s in the real-world, we match Jack’s collection one-to-one with Jill’s collection:

Since the leftover items are in Jill’s collection, the relationship between Jack’s collection and Jill’s collection is that:

Jack’s collection is smaller-in-size-than Jill’s collection

- When the leftover objects are in the first collection, we will say that the first collection is larger-in-size-than the second collection.

**Example 10.** To compare in the real-world Jack’s with Jill’s, we match Jack’s collection one-to-one with Jill’s collection:

Since the leftover items are in Jack’s collection, the relationship between Jack’s collection and Jill’s collection is that:

Jack’s collection is larger-in-size-than Jill’s collection

The relationship is the same as and the two strict relationships, is-smaller-than and is-larger-than, are mutually exclusive in the sense that as soon as we know that one of them holds, we know that neither one of the other two can hold.

3. Quite often, though, instead of the above three relationships, we will need to use another two relationships that we shall call **lenient**.
a. Instead of wanting to make sure that a first collection is-smaller-than a second collection, we may just want to make sure that the first collection is-no-larger-than the second collection, that is we may include collections that are-the-same-as.

What this mean is that instead of requiring that, after the one-to-one matching, the leftover items be in the second collection, we only require that the leftover items not be in the first collection and this is of course the case when the leftover items are in the second collection as before . . . but also when there are no leftover items in either collection and therefore certainly no leftover in the first collection.

**Example 11.** If Jack’s collection is and Jill’s collection is , then we have that:

Jack’s collection is no-larger-in-size-than Jill’s collection since, after one-to-one matching,

there is no leftover item in Jack’s collection.

**Example 12.** If Mike’s collection is and Jill’s collection is , it is also the case that:

Mike’s collection is no-larger-in-size-than Jill’s collection since, after one-to-one matching,

there is no leftover item in either collection and therefore certainly no leftover item in Mike’s collection.

b. Similarly, instead of wanting to make sure that a first collection is-larger-than a second collection, we may just want to make sure that the first collection is-no-smaller than the second collection, that is we include collections that are-the-same.

What this mean in the real-world is that instead of requiring that, after the one-to-one matching, the leftover items be in the first collection, we only require that the leftover items not be in the second collection and this
is of course the case when the leftover items are in the first collection as before ... but also when there are no leftover items in either collection and therefore certainly no leftover in the second collection.

**Example 13.** If Dick’s collection is \[\text{\$10, \$20, \$50, \$100, \$200}\] and Jane’s collection is \[\text{\$10, \$20, \$50, \$100, \$200}\], then we have that:

Dick’s collection is *no-smaller-in-size-than* Jane’s collection since, after one-to-one matching,

there is no leftover item in Jane’s collection.

**Example 14.** If Mary’s collection is \[\text{\$10, \$20, \$50, \$100, \$200}\] and Jane’s collection is \[\text{\$10, \$20, \$50, \$100, \$200}\], it is also the case that:

Mary’s collection is *no-smaller-in-size-than* Jane’s collection since, after one-to-one matching,

there is no leftover item in *either* collection and therefore certainly no leftover item in Jane’s collection.

The two *lenient* relationships are *not* mutually exclusive in the sense that, given two collections, even if we know that one *lenient* relationship is holding from the first collection to the second collection, we cannot be sure that the other *lenient* relationship does *not* hold from the first collection to the second collection because the first collection could be holding because the first collection is-the-same-as the second collection in which case the other *lenient* relationship would be holding too.

On the other hand, if *both* lenient relationships hold from a first collection to a second collection, then we know for sure that the first collection *is-the-same-as* the second collection.

### 2.3 Language For Comparisons

In order to *represent* on paper *relationships* between two collections, we first need to expand our *mathematical* language beyond *number-phrases*. 
1. Given a relationship between two collections, we need a verb to represent this relationship. In keeping with our distinguishing between what we do in the real-world and what we write on paper to represent it, as between a real-world process and the paper procedure that represents it, we use different words for a real-world relationships and for the verbs we write on paper to represent it:

   - To represent on paper the real-world *simple* relationships:
     - *is-the-same-in-size-as*, we will use the verb = which we will read as *is-equal-to*,
     - *is-different-in-size-from*, we will use the verb ≠ which we will read as *is-not-equal-to*,

   - To represent on paper the real-world *strict* relationships:
     - *is-smaller-in-size-than*, we will use the verb <, which we will read as *is-less-than*,
     - *is-larger-in-size-than*, we will use the verb > which we will read as *is-more-than*,

   - To represent on paper the real-world *lenient* relationships
     - *is-no-larger-in-size-than*, we will use the verb ≤, which we will read as *is-less-than-or-equal-to*,
     - *is-no-smaller-in-size-than*, we will use the verb ≥, which we will read as *is-more-than-or-equal-to*.

We will say that

   - The verbs > and < are *strict* verbs because they represent the *strict* relationships *is-smaller-in-size-than* and *is-larger-in-size-than*.
   - The verbs ≥ and ≤ are *lenient verbs* because they represent the *lenient* relationships *is-no-larger-in-size-than* and *is-no-smaller-in-size-than*.

2. Then, to indicate that a relationship holds from one collection to another, we write a *comparison-sentence* that consists of the *number-phrases* that represent the two collections with the verb that represents the relationship in-between the two number-phrases.

**Example 15.** Given Jack’s $3 and Jill’s $3, we represent the relationship  

>Jack’s collection is the same as Jill’s collection

by writing the *comparison-sentence*

>3 Dollars = 3 Dollars

which we read as

>THREE dollars is-equal-to THREE dollars.
EXAMPLE 16. Given Jack’s and Jill’s, we represent the relationship
Jack’s collection is different from Jill’s collection
by writing the comparison-sentence
\[ 3 \text{ Dollars} \neq 7 \text{ Dollars} \]
which we read as
THREE dollars is-not-equal-to SEVEN dollars.

EXAMPLE 17. Given Jack’s and Jill’s, we represent the relationship
Jack’s collection is different from Jill’s collection
by writing the comparison-sentence
\[ 5 \text{ Dollars} \neq 3 \text{ Dollars} \]
which we read as
FIVE dollars is-not-equal-to THREE dollars.

EXAMPLE 18. Given Jack’s and Jill’s, we represent the relationship
Jack’s collection is smaller than Jill’s collection
by writing the comparison-sentence
\[ 3 \text{ Dollars} < 7 \text{ Dollars} \]
which we read as
THREE dollars is less than SEVEN dollars.

EXAMPLE 19. Given Jack’s and Jill’s, we represent the relationship
Jack’s collection is larger than Jill’s collection
by writing the comparison-sentence
\[ 5 \text{ Dollars} > 3 \text{ Dollars} \]
which we read as
FIVE dollars is more than THREE dollars.

EXAMPLE 20. Given Jack’s and Jill’s, we rep-
sent the relationship
Jack’s collection is no-larger than Jill’s collection by writing the comparison-sentence
\[ 3 \text{ Dollars} \leq 5 \text{ Dollars}, \]
which we read as
THREE dollars is less-than-or-equal-to FIVE dollars.

**EXAMPLE 21.** Given Mike’s and Jill’s, we represent the relationship
Mike’s collection is no-larger than Jill’s collection by writing the comparison-sentence
\[ 5 \text{ Dollars} \leq 5 \text{ Dollars}, \]
which we read as
FIVE dollars is less-than-or-equal-to FIVE dollars.

**EXAMPLE 22.** Given Dick’s and Jane’s, we represent the relationship
Dick’s collection is no-smaller than Jane’s collection by writing the comparison-sentence
\[ 5 \text{ Dollars} \geq 2 \text{ Dollars}, \]
which we read as
THREE dollars is more-than-or-equal-to FIVE dollars.

**EXAMPLE 23.** Given Mary’s and Jane’s, we represent the relationship
Mary’s collection is no-smaller than Jane’s collection which we represent by writing the comparison-sentence
\[ 2 \text{ Dollars} \geq 2 \text{ Dollars}, \]
which we read as
THREE dollars is more-than-or-equal-to FIVE dollars.

3. Finally, comparison-sentences are named according to the verb that they involve
- Comparison-sentences involving the verb = are called **equalities**.
  **EXAMPLE 24.**
  \[ 3 \text{ Dollars} = 3 \text{ Dollars} \]
  is an equality
- Comparison-sentences involving the verb ≠ are called **plain inequalities**.
  **EXAMPLE 25.**
  \[ 3 \text{ Dollars} \neq 5 \text{ Dollars} \]
  is a plain inequality
2.4. PROCEDURES FOR COMPARING NUMBER-PHRASES

- Comparison-sentences involving the verbs $>$ or $<$ are called strict inequality (strict)
equalities.

**Example 26.**

$3 \text{ Dollars} < 7 \text{ Dollars}$ and $8 \text{ Dollars} > 2 \text{ Dollars}$ are strict inequality (strict) inequalities.

- Comparison-sentences involving the verbs $\leq$ and $\geq$ are called lenient inequality (lenient)
equalities.

**Example 27.**

$3 \text{ Dollars} \leq 7 \text{ Dollars}$ and $8 \text{ Dollars} \geq 2 \text{ Dollars}$ are lenient inequality (lenient) inequalities.

2.4 Procedures For Comparing Number-Phrases

Given two number-phrases, the procedure for writing the comparison-sentences that are true will depend on whether the number-phrases are basic counting number-phrases or decimal number-phrases.

Given two basic counting number-phrases, we must see whether we must count-up or count-down from the first numerator to the second numerator.¹

There are three possibilities depending on the direction we have to count when we count from the numerator of the first number-phrase to the numerator of the second number-phrase:

- We may have to count up, in which case the comparison-sentence is:

  first counting number-phrase $<$ second counting number-phrase
  (with $<$ read as “is-less-than”)

**Example 28.** To compare the given basic counting number-phrases 3 Washingtons and 7 Washingtons

i. We must count from 3 to 7:

$$4, 5, 6, 7$$

that is we must count up.

ii. So, we write the strict inequality:

$$3 \text{ Washingtons} < 7 \text{ Washingtons}$$

- We may have to count down, in which case the comparison-sentence is:

  first counting number-phrase $>$ second counting number-phrase
  (with $>$ read as “is-more-than”)

**Example 29.** To compare the given basic counting number-phrases 8 Washingtons and 2 Washingtons

i. We must count from 8 to 2:

$$7, 6, 5, 4, 3, 2$$

¹Educologists will be glad to know that, already in 1905, Fine was using the cardinal aspect for comparison processes in the real world and the ordinal aspect for comparison procedures on paper.
true
false

that is, we must count down.
ii. So, we write the strict inequality:

8 Washingtons > 2 Washingtons

• We may have neither to count up nor to count down, in which case the comparison-sentence is:

first counting number-phrase = second counting number-phrase
(with = read as “is-equal-to”)

EX A MPLE 30. To compare the given basic sentences 3 Washingtons and 3 Washingtons.
i. We must count from 3 to 3, that is we must count neither up nor down.
ii. So, we write the equality:

3 Washingtons = 3 Washingtons

2.5 Truth Versus Falsehood

Inasmuch as the comparison-sentences that we wrote until now represented relationships between real-world collections, they were true.

However, there is nothing to prevent us from writing comparison-sentences regardless of the real-world. In fact, there is nothing to prevent us from writing comparison-sentences that are false in the sense that there is no way that anyone could come up with real-world collections for which one-to-one matching would result in the relationship represented by these comparison-sentences.

EX A MPLE 31. The sentence

5 Dollars < 3 Dollars

is false because there is no way that anyone could come up with real-world collections for which one-to-one matching would result in there being leftover items in the second collection.

EX A MPLE 32. The sentence

5 Dollars = 3 Dollars,

is false because there is no way that anyone could come up with real-world collections for which one-to-one matching would result in there being no leftover item.

EX A MPLE 33. The sentence

3 Dollars ≤ 3 Dollars,

is true because we can come up with real-world collections for which one-to-one matching would result in there being no leftover item.

EX A MPLE 34. The sentence

5 Dollars ≤ 3 Dollars,
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is *false* because there is no way that anyone could come up with real-world collections for which *one-to-one matching* would result in there being leftover items in the second collection or *no* leftover item.

However, while occasionally useful, it is usually not very convenient to write sentences that are *false* because then we must not forget to write that they are false when we write them and we may miss that it says somewhere that they are false when we read them. So, inasmuch as possible, we shall write only sentences that are *true* and we will use

**DEFAULT RULE # 2.** *When no indication of truth or falsehood is given, mathematical sentences will be understood to be true and this will go without saying.*

When a sentence is *false*, rather than writing *it* and say that it is *false*, what we shall usually do is to write *its negation*—which is *true* and therefore “goes without saying”. We can do this either in either one of two manners:

- We can place the *false* sentence within the symbol \[ NOT[ \] \],
- We can just *slash* the *verb* which is what we shall usually do.

**EXAMPLE 35.** Instead of writing that

\[ \text{the sentence } 5 \text{ Dollars} = 3 \text{ Dollars is } \text{false} \]

we can either write the sentence

\[ NOT[5 \text{ Dollars} = 3 \text{ Dollars}] \]

or the sentence

\[ 5 \text{ Dollars} \neq 3 \text{ Dollars} \]

### 2.6 Duality Versus Symmetry

The *linguistic duality* that exists between `<` and `>` must not be confused with *linguistic symmetry*, a concept which we tend to be more familiar with\(^2\).

1. **Linguistic symmetry** involves pairs of sentences—which may be *true* or *false*—that represent *opposite* relationships between the two people/collections because, even though the verbs are *the same*, the two people/collections are mentioned in *opposite* order.

**EXAMPLE 36.**

\(^2\)This confusion is a most important *linguistic* stumbling block for students and one that Educologists utterly fail to take into consideration.
dual

- Jack is a child of Jill versus Jill is a child of Jack
- Jill beats Jack at poker versus Jack beats Jill at poker
- Jack loves Jill versus Jill loves Jack
- 9 Dimes > 2 Dimes versus 2 Dimes > 9 Dimes

Observe that just because one of the two sentences is true (or false) does not, by itself, automatically force the other to be either true or false and that whether or not it does depends on the nature of the relationship.

2. Linguistic duality involves pairs of sentences—which may be true or false—that represent the same relationship between the two people/collections because, even though the people/collections are mentioned in opposite order, the two verbs are dual of each other which “undoes” the effect of the order so that only the emphasis is different.

**Example 37.**
- Jack is a child of Jill versus Jill is a parent of Jack
- Jill beats Jack at poker versus Jack is beaten by Jill at poker
- Jack loves Jill versus Jill is loved by Jack
- 9 Dimes > 2 Dimes versus 2 Dimes < 9 Dimes

Observe that here, as a result, if one of the two sentences is true (or false) this automatically forces the other to be true (or false) and this regardless of the nature of the relationship.

3. When the verbs are the same and the order does not matter for these verbs, the sentences are at the same time (linguistically) symmetric and (linguistically) dual.

**Example 38.**
- Jack is a sibling of Jill versus Jill is a sibling of Jack
- 2 Nickels = 1 Dime versus 1 Dime = 2 Nickels

Observe that, here again, as soon as one sentence is true (or false), by itself this automatically forces the other to be true (or false) and that it does not depend on the nature of the relationship.