Chapter 5

Signed Number-Phrases

We have seen in Chapter 1 that we can use plain number-phrases, that is either counting number-phrases or decimal number-phrases, only in situations where the items are all of the same one kind. We shall now introduce and discuss a new type of number-phrase that we shall use in a type of situations that occurs frequently in which the items are all of either one of two kinds.

Just as we did for plain number-phrases in Chapters 2, 3, and 4, we will have to define for this new type of number-phrase what we mean by:

i. To “compare” two number-phrases,
ii. To “add” a second number-phrase to a first number-phrase,
iii. To “subtract” a second number-phrase from a first number-phrase.

and in particular to develop the corresponding procedures.

What will complicate matters a little bit, though, is that the procedures for the new type of number-phrases will involve the procedure that we developed for plain number-phrases. So, until we feel completely comfortable with the distinction, we shall use new symbols for “comparison”, “addition” and “subtraction” for the new kind of number-phrases.

5.1 Actions and States

Quite often we don’t deal with items that are all of the same kind but with items of two different kinds and a special case of this is when two items of different kinds cannot be together as they somehow cancel each other. As a result, we will now consider what we shall call two-way collections, that

1One can only wonder as to how Educologists can let their students use, without warning, the same symbols in these rather different situations.
is collections of items that are all of one kind or all of another kind with items of different kinds canceling each other.

1. In the real-world, two-way collections come up very frequently and in many different types of situations but they generally fall in either one of two types:
   - In one type of two-way collections, called actions, the items are steps in either this-direction or that-direction.

**Example 1.** In fact, we already encountered in the previous chapter this kind of items: counting up and counting down. Of course, the situation there was not symmetrical: we could always count steps up but we could not always count steps down. But there would have been no point counting at the same time three steps up and five steps down since steps up would cancel out steps down and this would have just amounted to counting two steps down.

**Example 2.**
   - Actions that a businesswoman may take on a bank account are to deposit three thousand dollars, withdraw two thousand dollars, etc
   - Actions that a gambler may take are to win fifty-eight dollars, lose sixty-two dollars, etc
   - Actions that a mark may take on a horizontal line include moving two feet leftward, five feet rightward, etc.
   - Actions that a mark may take on a vertical line include moving five inches upward, five inches downward, etc.

- In the other type of two-way collections, called states, the items are degrees of one kind or another but they have to be either on this-side or that-side of some benchmark.

**Example 3.**
   - States that a business may be in include being three thousand dollars in the red, being seven thousand dollars in the black, etc.
   - States that a gambler may be in include being sixty-two dollars ahead of the game, being thirty-seven dollars in the hole, etc.
   - States that a mark may be in on a horizontal line with some benchmark include being two feet to the left of the benchmark, being nine feet to the right of the benchmark, etc.
   - States that a mark may be in on a vertical line with some benchmark include being five inches above the benchmark, being three inches below the benchmark, etc.

2. Since all the items in a given two-way collection are of the same kind, a two-way collection is essentially a collection with a twist. So, just as we said that, in the real world,
   - the nature of a collection is the kind of items in the collection,
   - the extent of a collection is the number of items in the collection,
we shall now say that:
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- the **nature of an action** is the *kind* of steps in the action and the **nature of a state** is the *kind* of degrees in which the state can be
- the **extent of an action** is the *number* of steps in the action and the **size of a state** is the *number* of degrees of the state.
- the **direction of an action** is the *direction* of the steps in the action and the **side of a state** is the *side* of the degrees in the state.

**EXAMPLE 4.** When a person climbs up and down a ladder, an *action* may be climbing up seven rungs. Then,
- the **nature** of the action is **climbing rungs**
- the **size** of the action is **seven**
- the **direction** is **up**

## 5.2 Signed Number-Phrases

Plain number-phrases are not sufficient to represent on paper either *actions* or *states* because they do not indicate the *direction* of the action or the *side* of the state.

**EXAMPLE 5.**
- 3000 Dollars does not say if the businesswoman made a deposit or a withdrawal or if the business is in the red or in the black.
- 62 Dollars does not say if the gambler is ahead of the game or in the hole.
- 2 Feet does not say if the mark is to the left or to the right of the benchmark.
- 5 Inches does not say if the mark is moving up or down.

1. Since a two-way collection is just a collection with a *direction* or a *side*, we will represent on paper a two-way collection by a **signed number-phrase** that will consist of:
   - a **denominator** to represent on paper the **nature** of the action (that is the *kind* of the steps in the action) or of the state (that is the *kind* of the degrees in the state).
   - a **numerator** to represent on paper the **extent** of the action (that is the *number* of steps in the action) or the **extent** of the state (that is the *number* of degrees in the state),
   - a **sign** to represent on paper the **direction** of the action (that is the *direction* of the steps in the action) or the **side** of the state (that is the *side* of the benchmark that the degrees of the state are on.)

2. However, in order to say what direction the action or what side the state, we must always begin by **recording** for future reference:
   - which direction is to be the **standard direction** and which direction is therefore to be the **opposite direction**,
• which side of the benchmark is going to be the **standard side** and which side is therefore to be the **opposite side**.

**NOTE.** Historically, it has long gone without saying that *standard* was what was “good” and *opposite* what was “bad”.

**EXAMPLE 6.**
- To *deposit* money is usually considered to be “good” as it goes with *saving* while to *withdraw* money is usually considered to be “bad” as it goes with *spending*.
- To *win* is usually considered to be “good” while to *lose* is considered to be “bad”.
- To go *up* is usually considered to be “good” while to go *down* is usually considered to be “bad”.

3. Once we have recorded what is *standard* and therefore what is *opposite*, we can use a **sign** to represent on paper the *direction* of the action (that is the direction of the steps in the action) or the *side* of the state (that is the side of the benchmark that the degrees of the state are on):

   • we will use the sign +, read here as **positive**, to represent on paper whatever is *standard*, whether an action or a state.
   • we will use the sign −, read here as **negative**, to represent on paper whatever is *opposite*, whether an action or a state.

**NOTE.** This use of the symbols + and − is entirely different from their use in Chapter 1 where they denoted *addition* and *subtraction*. This complicates *reading* the symbol as we need to rely on the **context**, that is the text that is around the symbol, to decide what the symbol stands for.

4. However, because this will make developing and using *procedures* a lot easier, we will lump the **sign** together with the **numerator** and call the result a **signed-numerator**. Signed-numerator with a + are said to be **positive numerators** and signed-numerators with a − are said to be **negative numerators**.

**NOTE.** Historically, just as with standard and opposite and perhaps as a result, *positive* has been identified with “good” and *negative* with “bad”.

So, altogether, a **signed** number-phrase will consist of:

  • a signed-numerator
  • a denominator

**EXAMPLE 7.** Say that we have put on record that the **standard** direction is to *win* money so that to *lose* money is the **opposite** direction. Then,

<table>
<thead>
<tr>
<th>When a real-world gambler:</th>
<th>We write on paper:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>wins</strong> forty-seven dollars</td>
<td>+47 Dollars</td>
</tr>
<tr>
<td><strong>loses</strong> sixty-two dollars</td>
<td>−62 Dollars</td>
</tr>
</tbody>
</table>

**EXAMPLE 8.** Say we have put on record that the **standard** side is *in-the-black* so that *in-the-red* is the **opposite** side. Then,
5.3. **SIZE AND SIGN**

When a *real-world* business is: We write on *paper*:

<table>
<thead>
<tr>
<th>Description</th>
<th>Sign, of the numerator</th>
<th>Size, of the numerator</th>
</tr>
</thead>
<tbody>
<tr>
<td>three thousand dollars <em>in-the-black</em></td>
<td>+3000 Dollars</td>
<td></td>
</tr>
<tr>
<td>seven hundred dollars <em>in-the-red</em></td>
<td>−700 Dollars</td>
<td></td>
</tr>
</tbody>
</table>

5. We are using the same symbol, 0, both for
- the counting numerator that is left of the succession of counting numerators 1, 2, 3, 4, ...
- the signed numerator which is inbetween the succession of positive numerators +1, +2, +3, +4, ... and the recession of negative numerators −1, −2, −3, −4, ...

In this case, we shall have to live with the ambiguity and decide each time, according to the context, which one the numerator 0 really is.

### 5.3 Size And Sign

On the other hand, given a *signed numerator*, we shall say that:
- the **sign of the numerator** is the sign which was put in front of the plain numerator to make the signed numerator
- the **size** of the numerator is the plain numerator from which the signed numerator was made.

**Example 9.**

\[
\text{Signed Numerator} = -5
\]

\[
\text{Sign of Signed Numerator} = \\
\text{Size of Signed Numerator} =
\]

In other words, −5 is a signed-numerator whose *size* is 5 and whose *sign* is −.

**Example 10.**

\[
\text{Signed Numerator} = +3
\]

\[
\text{Sign of Signed Numerator} = \\
\text{Size of Signed Numerator} =
\]

In other words, +3 is a signed-numerator whose *size* is 3 and whose *sign* is +.

Indeed, signed number-phrases can contain more information than is necessary for a particular purpose and then all we need is either the *sign* or the *size* of the signed number-phrase.
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1. In many circumstances, what matters is only the size of the signed number-phrases and not the sign.

**Example 11.** Say we are told that

- Jill’s balance is $+70,000,000$ Dollars
- Jack’s balance is $−70,000,000$ Dollars.

We can safely conclude that neither Jack nor Jill belongs to “the rest of us”.

**Example 12.** If we are stopped on the turnpike doing $+100$ Miles Hour, that is while driving from Philadelphia to New York, or doing $−100$ Miles Hour that is while driving back from New York to Philadelphia, it does not matter which way we were going: regardless of the direction, we are going to get into big trouble.

So, in such cases, it is the size of the given signed numerator that matters.

**Example 13.** The size of Jill’s $+70,000,000$ Dollars is $70,000,000$ and the size of Jack’s $−70,000,000$ Dollars is also $70,000,000$ Dollars.

So, what makes Jack and Jill different from “the rest of us” is the size of their balance and not its sign.

**Example 14.** The size of our speed when we are going $+100$ Miles Hour (that is from Philadelphia to New York) is $100$ Miles Hour and the size of our speed when we are going $−100$ Miles Hour (that is from New York to Philadelphia) is also $100$ Miles Hour.

So, what gets us into trouble is the size of our speed.

2. In many other circumstances, what matters is only the sign of the signed number-phrase and not the numerator.

**Example 15.** Usually, banks do not accept negative balances, regardless of their size. In other words, all bank care about is the sign of the balance.

**Example 16.** If we are stopped going the wrong way on a one way street, it won’t matter if we were well under the speed limit. In other words, what gets us into trouble is the sign of our speed and not its size.

5.4 Graphic Illustrations

To graph a two-way collection represented on paper by a signed number-phrase, we proceed essentially just as with counting number-phrases and/or decimal number-phrases. The only differences are that on a signed ruler:

- we shall have the symbol for minus infinity, $−∞$, and the symbol for plus infinity, $+∞$, at the corresponding ends of the ruler

\begin{center}
\begin{tikzpicture}
  \draw[->] (-4,0) -- (4,0);
  \node at (-4,0) {$−∞$};
  \node at (4,0) {$+∞$};
  \node at (0,0) {$0$};
\end{tikzpicture}
\end{center}

- the tick-marks, if any, are labeled with signed number-phrases.

As with all rulers and depending on the circumstances, $0$ may or may not appear.

**Example 17.**
5.5. **Comparing Signed Number-Phrases**

We investigate the *first* fundamental process involving *actions* and *states*: Given two *actions* or two *states* we would like to be able to *compare* the signed number-phrases that represent them.

However, there are actually *two* viewpoints from which to compare signed number-phrases.

1. From what we shall call the **algebraic viewpoint**, the comparison depends both on the *sign* and the *size* of the two signed number-phrases. In the real-world, the comparison corresponds to the relationship is-smaller-than understood as is-poorer-than extended to the case when being in debt is allowed.

   It is traditional to use the same *verbs* as with counting number-phrases and decimal number-phrases, that is: $<$, $>$, $=$, and $\leq$, $\geq$.

   a. There are two cases depending on the signs of the two signed number-phrases:

      - When the signs of the two signed number-phrases are *the same*
        - any two positive number-phrases **algebra-compare** the same way as their *sizes* compare

**Example 22.**
algebra-more-than
algebra-less-than
is-left-of
is-right-of
size viewpoint

+365.75 Dollars > +219.28 Dollars
because 365.75 > 219.28.
– any two negative number-phrases algebra-compare the way opposite to the way their sizes compare

**Example 23.**

−432.69 Dollars < −184.41 Dollars
because 432.69 > 184.41.

• When the sign of the two signed number-phrases are opposite, we can say *either* that
  – any positive number-phrase is algebra-more-than any negative number-phrase
  or, *dually*, that
  – any negative number-phrase is algebra-less-than any positive number-phrase

**Example 24.**

−2386.77 Dollars < +17.871 Dollars
because any negative number-phrase is less-than any positive number-phrase.

b. In other words, when we *picture* on a ruler the signed number-phrases involved in an *algebraic comparison*, an algebraic comparison is about the relative positions of the two signed number-phrases relative to each other:

• *is-algebra-less-than* is pictured as *is-left-of*
• *is-algebra-more-than* is pictured as *is-right-of*

**Example 25.**

• The algebra-comparison sentence

  −4 Dollars < +2 Dollars

corresponds to the fact that in the graphic

  ![Graphic of a number line with marks at −4, 0, and 2] (The graphic shows a number line with marks at −4, 0, and 2, where the mark at −4 is left of the mark at 2.)

  the mark that represents −4 *is-left-of* the mark that represents +2

• The algebra-comparison sentence

  −1 Dollars > −4 Dollars

corresponds to the fact that in the graphic

  ![Graphic of a number line with marks at −4, −1, 0, and 2] (The graphic shows a number line with marks at −4, −1, 0, and 2, where the mark at −1 is right of the mark at −4.)

  the mark that represents −1 *is-right-of* the mark that represents −4

This illustrates the reason that we can reuse the same verbs with signed number-phrases as we did with counting number-phrases and decimal number-phrases.

2. From what we shall call the *size viewpoint*, the comparison depends
5.5. **COMPARING SIGNED NUMBER-PHRASES**

only on the *size* of the two signed number-phrases and *not* on the *sign.*

a. It is quite usual in the real-world to say that a hundred dollar debt is larger than a fifty dollar debt even though someone owing a hundred dollars is-poorer-than a person owing fifty dollars.

So, we will say that:

- A first signed number-phrase is-larger-in-size-than a second signed number-phrase when the size of the first signed number-phrase is larger than the size of the second signed number-phrase.

or, dually, we can say

- A first signed number-phrase is-smaller-in-size-than a second signed number-phrase when the size of the first signed number-phrase is smaller than the size of the second signed number-phrase.

We shall not use *symbols* and we shall just write the words.

**EXAMPLE 26.** We have of course that

\[+365.75 \text{ Dollars} \text{ is-larger-in-size-than} + 219.28 \text{ Dollars}\]

which corresponds to the fact that 365.75, the size of the first signed number-phrase, is larger than 219.28, the size of the second signed number-phrase.

We also have that

\[-365.75 \text{ Dollars} \text{ is-larger-in-size-than} − 219.28 \text{ Dollars}\]

which corresponds to the fact that 365.75, the size of the first signed number-phrase, is larger than 219.28, the size of the second signed number-phrase.

And we also have that

\[-365.75 \text{ Dollars} \text{ is-larger-in-size-than} + 219.28 \text{ Dollars}\]

which corresponds to the fact that 365.75, the size of the first signed number-phrase, is larger than 219.28, the size of the second signed number-phrase.

None of this has anything to do with the fact that, from the *algebra viewpoint,*

\[+365.75 \text{ Dollars} > +219.28 \text{ Dollars}\]

\[−365.75 \text{ Dollars} < −219.28 \text{ Dollars}\]

\[−365.75 \text{ Dollars} < +219.28 \text{ Dollars}\]

b. In other words, when we *illustrate* on a ruler the signed number-phrases involved in a *size comparison,* the comparison is about which numerator is-farther-away-from-the-center.

**EXAMPLE 27.**

- The size-comparison sentence

\[-4 \text{ Dollars} \text{ is-larger-in-size-than} + 1 \text{ Dollars}\]

corresponds to the fact that in the graphic

the mark that represents −4 Dollars is *farther-away-from-the-center-than* the mark that represents +1 Dollars.
follow up
merge
adding

• The size-comparison sentence

\[-4 \text{ Dollars is-larger-in-size-than } -3 \text{ Dollars}\]

corresponds to the fact that in the graphic

\[\text{Dollars}\]

the mark that represents \(-4\) farther-away-from-the-center-than the mark that represents \(-3\)

5.6 Adding a Signed Number-Phrase

We investigate the second fundamental process involving actions and states.

1. Just as in in the case of collections we could attach a second collection to a first collection, here we can

• follow up a first action with a second action.

**Example 28.**

- a gambler may win forty-five dollars and then follow up with winning sixty-two dollars.
- a gambler may win thirty-one dollars and then follow up with losing forty-four dollars.
- a gambler may lose twenty-one dollars and then follow up with winning fifty-seven dollars.
- a gambler may lose seventy-eight dollars and then follow up with losing thirty-four dollars.

• merge a first state with a second state

**Example 29.**

- a business that is three thousand dollars in the black may merge with a business that is six hundred dollars in the black.
- a business that is three hundred dollars in the black may merge with a business that is five hundred dollars in the red.
- a business that is two thousand dollars in the red may merge with a business that is seven hundred dollars in the black.
- a business that is seven hundred dollars in the red may merge with a business that is two hundred dollars in the red.

**NOTE.** English forces us to use a different word order here: while we attached a second collection to a first collection, here we must say that we follow up a first action with a second action. In order to be consistent, and although it is not necessary, we will also say that we merge a first state with a second state.

2. Then, just like adding a counting-number-phrases was the paper procedure to get the result of attaching a collection, adding a signed number-phrase will be the paper procedure to get the result of following up an action.
and/or merging a state.

In order to distinguish adding signed number-phrases from adding counting number-phrases as we develop the procedure, we shall use for a while the symbol \( \oplus \). Later, we will just use \(+\) and learn to rely on the context.

3. Just like, in Chapter 1, we introduced counting number-phrases with slashes, /, to discuss addition of signed number-phrases, we will use temporarily arrows of two kinds, \( \leftarrow \) and \( \rightarrow \).

**Example 30.** We will use temporarily
\[ \rightarrow \rightarrow \rightarrow \rightarrow \text{ Dollars} \] instead of \(+5\) Dollars
and
\[ \leftarrow \leftarrow \leftarrow \leftarrow \text{ Dollars} \] instead of \(-5\) Dollars.

When adding a signed number-phrase, we must distinguish two cases.

- a. The second signed number-phrase has the same sign as the first signed number-phrase. Then, all the items are of the same kind and so following up is the same as attaching. So, in that case, to get the size of the result, we add the sizes of the two signed number-phrases.

**Example 31.**

In the real-world, when we:
- deposit five dollars \( \rightarrow \rightarrow \rightarrow \rightarrow \text{ Dollars} \)
- then
- deposit three dollars, \( \rightarrow \rightarrow \text{ Dollars} \)
- altogether
- this \( \rightarrow \rightarrow \rightarrow \rightarrow \oplus \rightarrow \rightarrow \rightarrow \text{ Dollars} \)
- is the same as
- when \( \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \text{ Dollars} \)
- we
- deposit eight dollars \( \rightarrow \rightarrow \rightarrow \rightarrow \oplus \rightarrow \rightarrow \rightarrow \rightarrow \text{ Dollars} \)

or

**Example 32.**

In the real-world, when we:
- withdraw five dollars \( \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \text{ Dollars} \)
- then
- withdraw three dollars, \( \leftarrow \leftarrow \text{ Dollars} \)
- altogether
- this \( \leftarrow \leftarrow \leftarrow \leftarrow \oplus \leftarrow \leftarrow \leftarrow \text{ Dollars} \)
- is the same as
- when \( \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \rightarrow \rightarrow \text{ Dollars} \)
- we
- withdraw eight dollars \( \leftarrow \leftarrow \leftarrow \leftarrow \oplus \leftarrow \leftarrow \leftarrow \rightarrow \rightarrow \text{ Dollars} \)

- b. The second signed number-phrase has the opposite sign from the first signed number-phrase. Then, the items are of the same kind and so
following up is the same as attaching. So, in that case, to get the size of the result, we add the sizes of the two signed number-phrases.

**Example 33.**

In the real-world, when we deposit three dollars and then withdraw five dollars, altogether this is the same as attaching. So (in that case) to get the size of the result, we add the sizes of the two signed number-phrases.

EXAMPLE 33.

We write on paper:

- deposit three dollars → → Dollars
- and then ⊕
- withdraw five dollars, ← ← ← ← Dollars
- altogether =

this is [ → → ⊕ ← ← ← ← ] Dollars
the same [ → → ### ### ← ← ← ← ] Dollars
as [ → ### ### ← ← ← ← ] Dollars
when [### ### ← ← ← ← ] Dollars
we just → ← Dollars
withdraw two dollars ← ← 2 Dollars

or

**Example 34.**

In the real-world, when we deposit three dollars and then withdraw five dollars, altogether this is the same as attaching. So (in that case) to get the size of the result, we add the sizes of the two signed number-phrases.

EXAMPLE 34.

We write on paper:

- deposit three dollars → → Dollars
- and then ⊕
- withdraw five dollars, ← ← ← ← Dollars
- altogether =

this is [ → → ⊕ ← ← ← ← ] Dollars
the same [ → → ### ### ← ← ← ← ] Dollars
as [ → ### ### ← ← ← ← ] Dollars
when [### ### ← ← ← ← ] Dollars
we just → ← Dollars
withdraw two dollars ← ← 2 Dollars

**Theorem 1.** To add signed-numerators:

- When the two signed number-phrases have the same sign,
  - We get the sign of the result by taking the common sign
  - We get the size of the result by adding the two sizes.

- When the two signed number-phrases have opposite signs, we must first compare the sizes of the two signed number-phrases and then
  - We get the sign of the result by taking the sign of the signed number-phrase whose size is larger,
  - We get the size of the result by subtracting the smaller size from the larger size.
EXAMPLE 35. To identify the specifying-phrase \((+3) \oplus (+5)\) and since \((+3)\) and \((+5)\) have the same sign, we proceed as follows:

- We get the sign of the result by taking the common sign which gives us +
- We get the size of the result by adding the sizes 3 and 8 which gives us 8

In symbols,

\[ (+3) \oplus (+5) = (+[3 + 5]) \]
\[ = (+8) \]

EXAMPLE 36. To identify the specifying-phrase \((+3) \oplus (-5)\) and since \((+3)\) and \((-5)\) have opposite signs, we must compare the sizes. Since \(3 < 5\),

- We get the sign of the result by taking the sign of the number-phrase with the larger size which gives us –
- We get the size of the result by subtracting the smaller size, 3, from the larger size, 5 which gives us 2

In symbols,

\[ (+3) \oplus (-5) = (-[5 - 3]) \]
\[ = (-2) \]

5.7 Subtracting a Signed Number-Phrase

We investigate the third fundamental process involving actions and states.

While, in the case of collections, detaching a collection made immediate sense as “un-attaching”, in the case of actions “un-following up” and in the case of states “un-merging” do not make immediate sense. So, instead, we shall look at subtraction from the point of view of correction after we have done a long string of signed-additions and realized that there is an incorrect entry, that is a signed number-phrase that we shouldn’t have added (for whatever reason), so that the total is incorrect.

1. Up front, things would seem to work out exactly as in the case of un-signed number-phrases.

EXAMPLE 37. Suppose that we work in a bank and that we had added transactions as the day went which gave us the following specifying phrase

\[-2 \text{ Dollars} \oplus -7 \text{ Dollars} \oplus +5 \text{ Dollars} \oplus \ldots \oplus +3 \text{ Dollars}\]

and that at the end of day we identified the specifying-phrase which gave us

\[-132 \text{ Dollars}\]
but that we then realized that $-7$ Dollars was an outcast (it was not for a transaction but for money involved in some other matter) with the consequence that $-132$ Dollars is incorrect in that it is not the sum total of the transaction for the day.

2. To get the correct total, we have the following two choices for the procedure:

- **Procedure A** would be to strike out the incorrect signed number-phrase and redo the entire addition:

  **Example 38.** In the above example, we would strike out the incorrect entry $-7$ Dollars
  
  $-2$ Dollars $\oplus$ $-7$ Dollars $\oplus$ $+5$ Dollars $\oplus$ $\ldots$ $\oplus$ $+3$ Dollars
  
  Of course, since Procedure A is going to involve a lot of unnecessary work redoing all that had been done correctly, it is very inefficient.

- **Procedure B** would be to cancel out the effect of the incorrect entry on the incorrect total by subtracting the incorrect entry from the incorrect total.

  **Example 39.** In the above example, we would subtract the incorrect entry $-7$ Dollars from the incorrect total $-132$ Dollars
  
  $-132$ Dollars $\ominus$ $-7$ Dollars except that, at this point, we have no procedure for $\ominus$! Indeed, at this point, the only procedure we have for subtracting is for subtracting unsigned number-phrases.

  On the other hand, the obvious way to cancel out the effect of the incorrect entry on the incorrect total and that it is by adding the opposite of the incorrect entry to the incorrect total. (Accountants call this “entering an adjustment”.)

  **Example 40.** In the above example, we would add the opposite of the incorrect entry $-7$ Dollars, that is we would add $-7$ Dollars to the incorrect total $-132$ Dollars
  
  $-132$ Dollars $\oplus$ $+7$ Dollars

3. We now want to see that the two procedures must give us the same result either way. For that, we place the specifying-phrases in the two procedures side by side and we see that that the remaining number-phrases are the same either way.

  **Example 41.** In the above example, we place the specifying-phrases in the two procedures side by side:

  - The specifying-phrase in Procedure A is:
    
    $-2$ Dollars $\oplus$ $-7$ Dollars $\oplus$ $+5$ Dollars $\oplus$ $\ldots$ $\oplus$ $+3$ Dollars

  - The specifying-phrase in Procedure B is:
    
    $-2$ Dollars $\oplus$ $-7$ Dollars $\oplus$ $+5$ Dollars $\oplus$ $\ldots$ $\oplus$ $+3$ Dollars $\oplus$ $\mp$ $7$ Dollars

  We see that, either way, the remaining number-phrases are:

  $-2$ Dollars $\oplus$ $+5$ Dollars $\oplus$ $\ldots$ $\oplus$ $+3$ Dollars
5.8. EFFECT OF AN ACTION ON A STATE

4. Altogether then:
   - Adding the opposite of the incorrect entry (Procedure B):
     
     \[ -132 \text{ Dollars} \quad \oplus \quad +7 \text{ Dollars} \]

   necessarily amounts to exactly the same as
   - Striking out the incorrect entry (Procedure A):
     
     \[ -132 \text{ Dollars} \quad \ominus \quad -7 \text{ Dollars} \]

Since Procedure B is much faster than Procedure A, we say that the procedure for subtracting a signed number-phrase will be to add its opposite.

Example 42. In order to identify the specifying-phrase \((+3) \oplus (+5)\):
   1. we identify instead the specifying-phrase \((+3) \oplus (-5)\)
   2. we do the addition which gives us \(-2\)

Example 43. In order to identify the specifying-phrase \((-3) \oplus (-5)\):
   1. we identify instead the specifying-phrase \((-3) \oplus (+5)\)
   2. we do the addition which gives us \(+2\)

Example 44. In order to identify the specifying-phrase \((-3) \oplus (+5)\):
   1. we identify instead the specifying-phrase \((-3) \oplus (-5)\)
   2. we do the addition which gives us \(-8\)

Example 45. In order to identify the specifying-phrase \((+3) \oplus (-5)\):
   1. we identify instead the specifying-phrase \((+3) \oplus (+5)\)
   2. we do the addition which gives us \(+8\)

5.8 Effect Of An Action On A State

We now look at the connection between states and actions.

1. A state does not exist in isolation but is always one of many.

Example 46. The state of an account is usually different on different days. Given two states, we shall refer to the first one as the initial state and to the second one as the final state. The change from the initial state to the final state can be up in which case we shall call the change a gain or can be down in which case we shall call the change a loss.

   On paper, we shall use + for a gain and we shall use − for a loss.

Example 47.
   - At the beginning of a month, Jill’s account was two dollars in-the-red
• At the end of the month, Jill’s account was three dollars in-the-black.
So, during that month Jill’s account went up by five dollars and we shall write the gain as \(+5\) Dollars.

\[ \text{Change: } \text{+5 Dollars} \]

**States:**

\[ -\infty \rightarrow \star \rightarrow \star \rightarrow \star \rightarrow \star \rightarrow \star \rightarrow \star \rightarrow \star \rightarrow \star \rightarrow +\infty \]

**EXAMPLE 48.**
• At the beginning of a month, Jack’s account was two dollars in-the-black.
• At the end of the month, Jack’s account was five dollars in-the-red.
So, during that month Jack’s account went down by seven dollars and we shall write the loss as \(-7\) Dollars.

\[ \text{Change: } \text{-7 Dollars} \]

**States:**

\[ -\infty \rightarrow \star \rightarrow \star \rightarrow \star \rightarrow \star \rightarrow \star \rightarrow \star \rightarrow \star \rightarrow \star \rightarrow +\infty \]

**THEOREM 2.** Regardless of what the sign of the initial state and the sign of the final state are, we have that

\[ \text{change} = \text{final state } \oplus \text{ initial state} \]

**EXAMPLE 49.**
• At the beginning of a month, Jill’s account was two dollars in-the-red.
• At the end of the month, Jill’s account was three dollars in-the-black.

\[ \text{change} = +3 \text{ Dollars } \oplus -2 \text{ Dollars} \]

\[ = +3 \text{ Dollars } \oplus +2 \text{ Dollars} \]

\[ = +5 \text{ Dollars} \]

**EXAMPLE 50.**
• At the beginning of a month, Jack’s account was two dollars in-the-black.
• At the end of the month, Jack’s account was five dollars in-the-red.

\[ \text{change} = -5 \text{ Dollars } \oplus +2 \text{ Dollars} \]

\[ = -5 \text{ Dollars } \oplus -2 \text{ Dollars} \]

\[ = -7 \text{ Dollars} \]

2. A change always happens as the result of an action.

**EXAMPLE 51.** On an account,
• A deposit results in a gain.
• A withdrawal results in a loss.
In fact, we have exactly  

\[ \text{action} = \text{change} \]

so that, as a consequence of the previous THEOREM, actions and states are related as follows:

**THEOREM 3** (Conservation Theorem).

\[ \text{action} = \text{final state} \oplus \text{initial state} \]

**EXAMPLE 52.**
- On Monday, Jill’s account was five dollars *in-the-red*,
- On Tuesday, Jill *deposits* seven dollars.

So, we have:

i. 

\[ \text{Action} = +7 \text{ Dollars} \]

ii. 

\[ \text{States:} \]

\[ \text{Dollars} \]

So, on Wednesday, Jill’s account is two dollars *in-the-black*

iii. Then we compute the change:

\[ \text{Change} = \text{Final State} \oplus \text{Initial State} \]

\[ = +2 \text{ Dollars} \oplus -5 \text{ Dollars} \]

\[ = +2 \text{ Dollars} \oplus +5 \text{ Dollars} \]

\[ = +7 \text{ Dollars} \]

And we have indeed that

\[ \text{action} = \text{final state} - \text{initial state} \]

What happened is that each state is the result of all prior actions. So, by subtracting the initial state from the final state, we eliminate the effect of all the actions that resulted in the initial state, that is the effect of all the actions except the effect of the last one, namely the seven dollars deposit.

### 5.9 From Plain To Positive

We now have two kinds of number-phrases: *plain* number-phrases and *signed* number-phrases. The two, though, overlap and we want to analyze the connections between the two and what is gained when we go from using *plain* number-phrases to using *signed* number-phrases.
1. We developed
- *plain* number-phrases in order to deal with collections of items that are all of *one* kind,
- *signed* number-phrases in order to deal with collections of items that are all of one kind or all of another kind—with items of different kinds canceling each other.

But then, given collections of items that are all of *one* kind, it often happens that we can eventually think of another kind of items that cancel the first kind of items.

**Example 53.** We may start *counting* steps to find out *how much we walked*. But eventually, we may want to know *how far we progressed*, being that there are steps *backward* as well as step *forward* and, if it doesn’t matter what kind of steps they are when it comes to *how much we walked*, it does matter very much when it comes to *how far we progressed* and so we need to keep track of the direction of the steps.

2. But then, we can represent the original collection of items in two ways:
- With a *plain* number-phrase
- With a *positive* number-phrase

**Example 54.** Given a collection of seven steps (necessarily all in the same direction since all items in a collection have to be the same), we can represent the collection by:
- the plain number-phrase
  
  \[
  \begin{align*}
  7 \text{ Steps} \\
  \end{align*}
  \]

  - or we can adopt that direction as *standard direction* and then represent the collection by the *positive* number-phrase
  
  \[
  \begin{align*}
  +7 \text{ Steps} \\
  \end{align*}
  \]

3. We now check that, when we do an addition, we can go either one of two routes:
- We can first *replace* the two *plain* number-phrases by *positive* number-phrases and then *oplus* the two *positive* number-phrases,
- We can add the two *plain* number-phrases and then *replace* the result of the addition by a *positive* number-phrase.

Both routes get us to the same result.

**Example 55.**

This works also with *subtraction*. 
NOTE. The reader should check on her/his own that if, instead of replacing *plain* number-phrases by *positive* number-phrases, we were to replace *plain* number-phrases by *negative* number-phrases, then things would not always work in the sense that the two routes would not always result with the same number-phrase.