worth
unit-worth
value
unit-value

Chapter 6

Co-Multiplication and Values

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We seldom deal with a collection without wanting to know what the (money) **worth** of the collection is, that is how much money the collection could be exchanged for.

6.1 Co-Multiplication

Since all the items in a *collection* are the same, to find the *worth* of that collection, we need only know the **unit-worth** of the items, that is the amount of money that any one of these items can be exchanged for.

EXAMPLE 1. Given a collection of five apples, and given that the *unit-worth* of apples is seven cents, the real-world *process* for finding the *worth* of the collection is to exchange each apple for seven cents. Altogether, we end up exchanging the whole collection for thirty-five cents which is therefore the *worth* of the collection.

We now want to develop a paper *procedure* to get the number-phrase that represents the *worth* of the given collection, which we will call **value**, in terms of the number-phrase that represents the *unit-worth* of the items in the collection, which we will call **unit-value**.

1. We know how to write the number-phrase that represents the given *collection* and how to write its *value*, that is the number-phrase that represents its *worth*, but what is not obvious is how we should write the *unit-value* that is the number-phrase that represents the *unit-worth*.

EXAMPLE 2. In EXAMPLE 1, we represent the collection of five apples by writing the number-phrase 5 **Apples** and we represent its worth by writing its *value*, that is the number-phrase 35 **Cents**.
What is not obvious is how to write the *unit-value* of the **Apples**, that is the number-phrase that represents the *unit-worth* of the apples, that is the fact that "each apple is

co-denominator
co-multiplication

worth seven cents".

More specifically, we know what the *numerator* of the unit-value should be but what we don't know is how to write the *denominator* of the unit-value which we will call **co-denominator**.

Looking at the real-world shows that the *procedure* for finding the *value* must involve *multiplication* so that the *specifying-phrase* must look like:

Number-phrase for collection × *Unit-value* = Number-phrase for money

EXAMPLE 3. In EXAMPLE 2, the number-phrase that represents the *collection* is 5 **Apples** and the numerator of the unit-phrase that represents the unit-value of the items is 7 so the *specifying-phrase* must look like

$$5\ \textbf{Apples}\ \times\ 7\ ???$$

where ??? stands for the *co-denominator*.

2. The *co-denominator* should be such that the procedure for going from the specifying phrase to the result should prevent the denominator of the number-phrase for the *collection* from appearing in the *result* and, at the same time, be such as to force the denominator of the number-phrase for the *value* to appear in the *result*.

EXAMPLE 4. In EXAMPLE 3, since we must have

$$5\ \textbf{Apples}\ \times\ 7\ ??? = 35\ \textbf{Cents}$$

the *procedure* to go from the specifying phrase on the left, that is 5 **Apples** × 7 ???, to the result on the right, that is 35 **Cents**, must

- prevent **Apples** from appearing on the right
- but force **Cents** to appear on the right.

3. What we will do is to write the *co-denominator* just like a *fraction* with:

- the denominator of the *value above* the *bar*
- the denominator of the *items below* the *bar*.

EXAMPLE 5. In EXAMPLE 4, we write $\frac{\textbf{Cents}}{\textbf{Apple}}$ in place of ??? so that the specifying-phrase becomes

$$5\ \textbf{Apples}\ \times\ 7\ \frac{\textbf{Cents}}{\textbf{Apple}}$$

That way, the procedure for identifying such a specifying phrase, called **co-multiplication**, is quite simply stated:

i. multiply the *numerators*

ii. multiply the *denominators* with cancellation.

EXAMPLE 6. When we carry out the procedure on the specifying phrase in EXAMPLE 5, we get

$$5\ \textbf{Apples}\ \times\ 7\ \frac{\textbf{Cents}}{\textbf{Apple}} = (5 \times 7)\left(\cancel{\textbf{Apples}} \times \frac{\textbf{Cents}}{\cancel{\textbf{Apple}}}\right)$$
$$= 35\ \textbf{Cents}$$

co-number-phrase
evaluate
percentage
extend

which is what we needed to represent the real-world situation in EXAMPLE 1.

4. From now on, in order to remind ourselves that the reason why *unit-values* are written this way is to make it easy to *co-multiply*, we shall call them **co-number-phrases**[1].
Also, just as we often say "To *count* a collection" as a short for "To find the numerator of the number-phrase that represents a collection", we shall say "To **evaluate** a collection" as a short for "To find the numerator of the number-phrase that represents the *value* of a collection".

NOTE. Co-multiplication is at the heart of a part of mathematics called DIMENSIONAL ANALYSIS that is much used in sciences such as PHYSICS, MECHANICS, CHEMISTRY and ENGINEERING where people have to "cancel" denominators all the time.

EXAMPLE 7.

$$5\ \textsf{Hours} \times 7\ \frac{\textsf{Miles}}{\textsf{Hour}} = (5\times 7)\left(\cancel{\textsf{Hours}} \times \frac{\textsf{Miles}}{\cancel{\textsf{Hour}}}\right) = 35\ \textsf{Miles}$$

EXAMPLE 8.

$$5\ \textsf{Square-Inches} \times 7\ \frac{\textsf{Pound}}{\textsf{Square-Inch}} = (5\times 7)\left(\cancel{\textsf{Square-Inches}} \times \frac{\textsf{Pound}}{\cancel{\textsf{Square-Inch}}}\right) = 35\ \textsf{Pounds}$$

Co-multiplication is also central to a part of mathematics called LINEAR ALGEBRA that is in turn of major importance both in many other parts of mathematics and for all sort of applications in sciences such as ECONOMICS.

EXAMPLE 9.

$$5\ \textsf{Hours} \times 7\ \frac{\textsf{Dollars}}{\textsf{Hour}} = (5\times 7)\left(\cancel{\textsf{Hours}} \times \frac{\textsf{Dollar}}{\cancel{\textsf{Hour}}}\right) = 35\ \textsf{Dollars}$$

More modestly, *co-multiplication* also arises in **percentage** problems:

EXAMPLE 10.

$$5\ \textsf{Dollars} \times 7\ \frac{\textsf{Cents}}{\textsf{Dollar}} = (5\times 7)\left(\cancel{\textsf{Dollars}} \times \frac{\textsf{Cents}}{\cancel{\textsf{Dollar}}}\right) = 35\ \textsf{Cents}$$

6.2 Effect of Transactions on States: Signed Co-Multiplication

We now want to **extend** the concept of *co-multiplication* to *signed*-number-phrases in order to deal with *actions* and *states*.

[1]Educologists will of course have recognized number-phrases and co-number-phrases for the *vectors* and *co-vectors* that they are—albeit one-dimensional ones.

signed co-number-phrase

1. We begin by looking at the real-world. As before, we want to investigate the *change* in a given state, *gain* or *loss*, that results from a given transaction, "in" or "out" as before but with *two-way collections* of "good" items or "bad" items.

EXAMPLE 11. Consider a store where, for whatever reason best left to the reader's imagination, collections of apples can either get in or out of the store. Moreover, the collections are really two-way collections in that the apples can be either *good*—inasmuch as they will generate a sales profit—or *bad*—inasmuch as they will have to be disposed of at a cost.

2. We now look at the way we will represent things on paper.

a. To represent collections that can get *in* or *out*, we use *signed number-phrases* and we use a + sign for collections that get *in* and a − sign for *collections* that get *out*.

So, we will represent

- *collections* getting "in" by *positive* number-phrases,
- *collections* getting "out" by *negative* number-phrases,

EXAMPLE 12. In the above example, we would represent

- a collection of three apples getting *in* the store by the number-phrase $+3$ **Apples**
- a collection of three apples getting *out* of the store by the number-phrase -3 **Apples**

b. To represent unit-values that can be *gains* or *losses*, we use **signed co-number-phrase** and we use a + sign to represent *gains* and a − sign to represent *losses*.

So, we will represent

- the unit-value of *"good"* items by *positive* co-number-phrases,
- the unit-value of *"bad"* items by *negative* co-number-phrases,

EXAMPLE 13. In the above example, we would represent

- the unit-value of apples that will generate a sales *profit* of seven cents per apples by the co-number-phrase $+7\ \frac{\textbf{Cents}}{\textbf{Apple}}$
- the unit-value of apples that will generate a disposal *cost* of seven cents per apple by the co-number-phrase $-7\ \frac{\textbf{Cents}}{\textbf{Apple}}$

3. Looking at the *effect* that *transactions* (of two-way collections) can have on (money) *states*, that is at the fact that:

- A two-way collection of "good" items getting "in" makes for a "good" change.
- A two-way collection of "good" items getting "out" makes for a "bad" change.
- A two-way collection of "bad" items getting "in" makes for a "bad" change.
- A two-way collection of "bad" items getting "out" makes for a "good" change.

we can now write the procedure for **signed co-multiplication** for which we will use the symbol $\otimes$: signed co-multiplication

i. multiply the *denominators* (with cancellation).

ii. multiply the *numerators* according to the way gains and losses occur:

- $(+) \otimes (+)$ gives $(+)$

EXAMPLE 14.

Three apples get *in* the store. $+3$ **Apples**

The apples have a unit-value of seven cents-per-apple *gain*. $+7 \frac{\text{Cents}}{\text{Apple}}$

The specifying phrase is $[+3 \text{ Apples}] \otimes \left[+7 \frac{\text{Cents}}{\text{Apple}}\right]$

We co-multiply $[(+3) \otimes (+7)] \left[\cancel{\text{Apples}} \times \frac{\text{Cents}}{\cancel{\text{Apple}}}\right]$

We get a twenty-one cent *gain*. $= +21$ **Cents**

- $(+) \otimes (-)$ gives $(+)$

EXAMPLE 15.

Three apples get *in* the store. $+3$ **Apples**

The apples have a unit-value of seven cents-per-apple *loss*. $-7 \frac{\text{Cents}}{\text{Apple}}$

The specifying phrase is $[+3 \text{ Apples}] \otimes \left[-7 \frac{\text{Cents}}{\text{Apple}}\right]$

We co-multiply $[(+3) \otimes (-7)] \left[\cancel{\text{Apples}} \times \frac{\text{Cents}}{\cancel{\text{Apple}}}\right]$

We get a twenty-one cent *loss*. $= -21$ **Cents**

- $(-) \otimes (+)$ gives $(+)$

EXAMPLE 16.

Three apples get *out* of the store. -3 **Apples**

The apples have a unit-value of seven cents-per-apple *gain*. $+7 \frac{\text{Cents}}{\text{Apple}}$

The specifying phrase is $[-3 \text{ Apples}] \otimes \left[+7 \frac{\text{Cents}}{\text{Apple}}\right]$

We co-multiply $[(-3) \otimes (+7)] \left[\cancel{\text{Apples}} \times \frac{\text{Cents}}{\cancel{\text{Apple}}}\right]$

We get a twenty-one cent *loss*. $= -21$ **Cents**

- $(-) \otimes (-)$ gives $(+)$

EXAMPLE 17.

Three apples get *out* of the store. -3 **Apples**

The apples have a unit-value of seven cents-per-apple *loss*. $-7 \frac{\text{Cents}}{\text{Apple}}$

The specifying phrase is $[-3 \text{ Apples}] \otimes \left[-7 \frac{\text{Cents}}{\text{Apple}}\right]$

We co-multiply $[(-3) \otimes (-7)] \left[\cancel{\text{Apples}} \times \frac{\text{Cents}}{\cancel{\text{Apple}}}\right]$

We get a twenty-one cent *gain*. $= +21$ **Cents**

NOTE. The choice of symbols, + to represent *good* and − to represent *bad*, was not an arbitrary choice because of the way they interact with the symbols for *in* and *out*. We leave it as an exercise for the reader to investigate what happens when other choices are made.

4. Just as with *addition* and *subtraction*, in the case of *co-multiplication* too, we can replace *plain* number-phrases by *positive* number-phrases .

EXAMPLE 18.