Chapter 7

Basic Problems 1:
(Counting Numerators)

In the real world, we often select collections on the basis of requirements that these collections must meet. After introducing some more mathematical language and discussing real-word situations, we will develop a paper world approach and introduce what will be our general procedure when dealing with such problems.

7.1 Forms, Data Sets And Solution Subsets

We begin by looking at the way we deal in English with the selection of collections in the real world.

1. Essentially, what we use are “incomplete sentences” like those we encounter on certain exams or when we have to enter a noun in the blanks of a form.

   EXAMPLE 1. The following

   ___________________ is a past President of the United States.

   is a form in which the box is the blank in which we are supposed to enter a noun.

   2. The instruction to enter some given noun in the blank of a form may result in:

      - nonsense, that is words that say nothing about the real world.

   EXAMPLE 2. The instruction to enter the data,

   Mathematics

   in the blank of the form
sentence
TRUE
FALSE
data set
curly brackets
{ }
problem

3. In order to avoid having to deal with nonsense, that is in order to

   make sure that when we enter a noun we always get a sentence, regardless

even whether that sentence turns out to be true or false, we will always have

data set from which to take the nouns.

   We shall write the data set by writing the data within a pair of curly
brackets { }

   EXAMPLE 4. Given the form

   the following could be a data set

   {Bill Clinton, Ronald Reagan, Jennifer Lopez, John Kennedy, Henry Ford}

   but the following could not be a data set

   {Bill Clinton, Ronald Reagan, Jennifer Lopez, Mathematics, Henry Ford}

4. A problem will consist of a form together with a data set.

   EXAMPLE 5. The form

   and the data set
7.2. COLLECTIONS MEETING A REQUIREMENT

{Bill Clinton, Ronald Reagan, Jennifer Lopez, John Kennedy, Henry Ford} make up a problem.

a. Given a problem, that is given a data set and a form,
   • a solution (of the given problem) is a noun such that, when we enter this noun into the blank of the form, the result in a sentence that is TRUE
   • a non-solution (of the given problem) is a noun such that, when we enter this noun into the blank of the form, the result in a sentence that is FALSE

**EXAMPLE 6.** Given the problem consisting of

the form

[ ] is a past President of the United States.

and the data set

{Bill Clinton, Ronald Reagan, Jennifer Lopez, John Kennedy, Henry Ford}

• The solutions of the problem are
  Bill Clinton, Ronald Reagan, John Kennedy
• The non-solutions of the problem are
  Jennifer Lopez, Henry Ford

b. Given a problem, that is given a data set and a form, the solution subset for the problem consists of all the solutions. We write a solution subset the same way as we write a data set, that is we write the solutions between brackets {  }.

**EXAMPLE 7.** Given the problem consisting of

the form

[ ] is a past President of the United States.

and the data set

{Bill Clinton, Ronald Reagan, Jennifer Lopez, John Kennedy, Henry Ford} the solution subset of that problem is

{Bill Clinton, Ronald Reagan, John Kennedy}

7.2 Collections Meeting A Requirement

The simplest way to select collections from a given set of selectable collections is to require them to compare in a given way to a given gauge collection which we do by matching the collections one-to-one with the gauge collection. (See Chapter 2.) The result is what we will call the select subset.

**EXAMPLE 8.** Jack has the following collection of one-dollar bills
So the bids that he can at all make in an auction (set of selectable collections) are:

If the starting bid (gauge collection) for a particular object is three dollars (a selectable collection), the bids that Jack could make (select subset) would then be:

1. The gauge collection may or may not be a selectable collection.

**Example 9.** Jack has the following collection of one-dollar bills

So the bids that he can at all make in an auction (set of selectable collections) are:

- If the starting bid for a particular object is three dollars (a selectable collection), then the bids that he could make (select subset) would be:

- If the starting bid for a particular object is three dollars and forty cents (not a selectable collection), then the bids that he could make (select subset) would be:

2. The way the selectable collections are required to compare with the gauge collection can be to be:
   - larger-in-size than the gauge collection,
   or
   - smaller-in-size than the gauge collection,
   or
   - same-in-size as the gauge collection.
   or
   - different-in-size from the gauge collection,
   or
   - no-larger-in-size than the gauge collection,
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- no-smaller-in-size than the gauge collection.

**EXAMPLE 10.** Jack has the following collection of one-dollar bills

So the bids that he can at all make in an auction correspond to the collections of one-dollar bills that he can use (set of selectable collections):

- If it is the starting bid for a particular object that is three dollars, then the bids that Jack could make (select subset) would be:

- If it is the current bid for a particular object that is three dollars, then the bids that Jack could make would be:

3. Occasionally, the subset of selected collections can be empty meaning that none of the selectable collections meets the given requirement.

**EXAMPLE 11.** Jack has the following collection of one-dollar bills

So the bids that he can at all make in an auction (set of selectable collections) are:

If the starting bid (gauge collection) for a particular object is seven dollars, then Jack cannot make any bid so that the select subset is empty.

4. Occasionally, the subset of selected collections can be full meaning that all of the selectable collections meet the given requirement.

**EXAMPLE 12.** Jack has the following collection of one-dollar bills

So the bids that he can at all make in an auction (set of selectable collections) are:

If the starting bid (gauge collection) for a particular object is one dollars, then the select subset is full.

There is of course nothing difficult with the one-to-one matching process involved in checking whether selectable collections compare or do not compare in a given way with a given gauge collection, but, as with most real-world processes, all this one-to-one matching of items is certainly going to
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get very quickly very tedious.

7.3 Basic Formulas

In order to represent on paper real-world the various situations involving
the selection on the basis of a requirement of a subset of selected collection
from among a set of selectable collections we will use:
• Number-phrases to represent the collections,
• The six verbs that were introduced in Chapter 2 to compare collections
  \( >, <, =, \neq, \leq, \geq \)
• A special kind of form to represent the requirement.
  1. The main difficulty with forms as we discussed them in Section 7.1
     above is with the blanks. So we begin by introducing a kind of form that
     will be appropriate for “computations”.
     a. Instead of blanks, we will use an unspecified-numerator such as,
        for instance, the letter \( x \).
        \textbf{Example 13.} Instead of writing
        \[
        \square < 5
        \]
        we will write
        \[
        x < 5
        \]
     b. A specifying-formula — we will often say formula for short—is
        a kind of forms in which:
        • the verb can be any one of:
          \( >, <, =, \neq, \leq, \geq \)
        • the nouns are numerators
        • the common denominator is factored out.
        \textbf{Example 14.} The following are specifying-formulas
        \[
        x \leq 8 \\
        x + 3 \geq 8 \\
        3 \times x < 12 \\
        -3 \otimes x \oplus -7 = -12
        \]
        We will distinguish between:
        • \textbf{Equations}, that is specifying-formulas that involve the verb
        \[
        =
        \]
        • \textbf{Inequalities}\footnote{Although supposedly exceedingly concerned with the relevance of mathematics to the “ordinary life” of their students—as opposed to their “school life” one can only suppose, but judging by the textbooks they produce in vast numbers, Educologists are strangely indifferent to the fact that, in the real world, inequalities are vastly more prevalent than equations.}, that is specifying-formulas that involve any one of the
other five verbs:  
\[ >, <, \neq, \leq, \geq \]

c. Then, instead of giving the **instruction** 

\[
\text{enter the given numerator in the blank.}
\]

we will give the **instruction**  

**replace** the unspecified numerator \( x \) by the **given** numerator.

**EXAMPLE 15.** Instead of giving the **instruction**  

Enter 7 in the **blank** of the **form**:

\[
\_ < 5.
\]

we will give the **instruction**  

**Replace** \( x \) by 7 in the **formula**:

\[
x < 5
\]

d. While a **formula** is **not a sentence** because it does not say anything about the real world (how could it since all that \( x \) stands for is a **blank**?), once we have replaced in a formula the unspecified numerator \( x \) by a given numerator, we have of course a **sentence**. (That this sentence is going to be either **TRUE** or **FALSE** depending on the given numerator is beside the point here.)

**EXAMPLE 16.** The **specifying-formula**

\[
x < 5
\]

is **not a sentence** because it does not say anything about the real world since \( x \) does not stand for a given numerator.

The **instruction** to **replace** \( x \) by 7 in the **specifying-formula**

\[
x < 5
\]

results in

\[
7 < 5
\]

which is a **sentence**. (That it happens to be **FALSE** is beside the point here.)

e. What will complicate matters a bit is that we will often **code** the **instruction** to replace the unspecified numerator \( x \) by some given numerator into the specifying-formula itself. For that, we will

i. **draw**, to the right of the specifying formula a **vertical bar** extending a bit **below** the line, which we read as “where”

ii. **write** to the bottom right of the **vertical bar**:

- the unspecified numerator \( x \)

followed by

- the symbol :=, to be read as “is to be replaced by”,

followed by

- the **given numerator**

\[
equations.
\]
Example 17. Instead of using the instruction
Replace $x$ by 7 in the specifying-formula:
\[ x < 5 \]
we shall write the instruction right into the specifying formula as follows:
\[ x < 5 |_{x=7} \]
and the result is to be read as:
\[ x < 5 \]
where $x$ is to be replaced by 7.
The reason this complicates matters is that while
\[ x < 5 \]
is a specifying-formula,
\[ x < 5 \]
is a sentence since it is the same as the sentence
\[ 7 < 5 \]
f. In particular, we have that:
- replacing the unspecified numerator by a given numerator in an inequation results in an inequality.

Example 18. Using a form, we would write: \( x > 3.14 \)
Using a formula, we will write:
\( x > 3.14 \)

<table>
<thead>
<tr>
<th>“Before”: Inequation (neither TRUE nor FALSE)</th>
<th>“Action”: Enter 7.82 in the blank (neither TRUE nor FALSE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( &gt; 3.14 )</td>
<td>( &gt; 3.14 )</td>
</tr>
<tr>
<td>7.82</td>
<td>Replace ( x ) by 7.82</td>
</tr>
<tr>
<td>( \times_{:=7} )</td>
<td>( \times_{:=7,82} )</td>
</tr>
</tbody>
</table>

Example 19. Using a form, we would write: \( x = +5 \)
Using a formula, we will write:
\( x = +5 \)

<table>
<thead>
<tr>
<th>“Before”: Equation (neither TRUE nor FALSE)</th>
<th>“Action”: Enter (-3) in the blank (neither TRUE nor FALSE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( = +5 )</td>
<td>( = +5 )</td>
</tr>
<tr>
<td>(-3)</td>
<td>Replace ( x ) by (-3)</td>
</tr>
<tr>
<td>( \times_{:=\text{FALSE}} )</td>
<td>( \times_{:=\text{FALSE}} )</td>
</tr>
</tbody>
</table>
2. Given a formula, the associated formulas for that formula are the formulas that differ from the given formula only by the verb.

Crucial for the general procedure that we will develop in the next chapter and given an inequation regardless of whether this given inequation is strict or lenient, are:

- The associated equation, that is the equation we obtain by replacing the verb in the given inequation by the verb =.

**Example 20.** The equation associated with the lenient inequation

\[-3 \otimes x \geq +90.43\]

is the equation

\[-3 \otimes x = +90.43\]

**Example 21.** The equation associated with the strict inequation

\[x \oplus -14.08 < +53.71\]

is the equation

\[x \oplus -14.08 = +53.71\]

- The associated strict inequation, that is the inequation we obtain by replacing the verb in the given inequation by the corresponding strict verb.

**Example 22.** Given the lenient inequation

\[x + 6.08 \geq 17.82\]

the associated strict inequation is

\[x + 6.08 > 17.82\]

So, the strict inequation associated to a strict inequation is the strict inequation itself.

**Example 23.** Given the strict inequation

\[x \ominus -6.08 < -44.78\]

the associated strict inequation is

\[x \ominus -6.08 < -44.78\]

While certainly surprising, this will help us developing a general procedure in the next chapter.

In particular, we can say that a lenient inequation gives the choice between the associated strict inequation and the associated equation.

**Example 24.** The lenient inequation

\[x \leq +53.71\]

gives the choice between the associated strict inequation:

\[x < +53.71\]

and the associated equation

\[x = +53.71\]

For instance,

-61.05 is a solution of \[x \leq +53.71\] because -61.05 is a solution of \[x < +53.71\]

and

+53.71 is a solution of \[x \leq +53.71\] because +53.71 is a solution of \[x = +53.71\]
3. The simplest kind of specifying formula, which we will call **basic formulas**, are formulas involving two *nouns* related by a *verb* in the following manner:

i. The first *noun* is the **unspecified numerator** $x$, 

ii. The *verb* is any of the verbs introduced in Chapter 2 to compare collections:

iii. The second *noun* is a given **gauge numerator**

**EXAMPLE 25.** The following specifying-phrases are basic formulas:

\[ x < 5 \]
\[ x \geq -3 \]
\[ x \neq -52.19 \]

but the following specifying phrases are not basic formulas:

\[ x + 3 \geq 8 \]
\[ 3 \times x < 12 \]
\[ 3 \otimes x \oplus -7 = -12 \]

In order to talk in general about basic formulas, we will use the symbol $x_0$ to stand for the **gauge numerator**.

4. We will sort basic formulas according to the kind of verb that is involved and we will distinguish four types of basic formula corresponding to the four types of *comparison sentences* that we encountered in Chapter 2.

- **Basic equations** are basic formulas of the type:

  \[ x = x_0 \]

  **EXAMPLE 26.** The formula
  
  \[ x = 31.19 \]

  is a **basic equation**

- **Basic simple inequations** are basic formulas of type:

  \[ x \neq x_0 \]

  **EXAMPLE 27.** The formula
  
  \[ x \neq 742.05 \]

  is a **basic simple inequation**

- **Basic strict inequations** are basic formulas of type:

  \[ x > x_0 \quad \text{or} \quad x < x_0 \]

  **EXAMPLE 28.** The formulas
  
  \[ x > 132.17 \]
  
  and
  
  \[ x < -283.41 \]

  are both **basic strict inequations**

- **Basic lenient inequations** are basic formulas of type:
EXAMPLE 29. The formulas
\[ x \leq x_0 \quad \text{or} \quad x \geq x_0 \]
and
\[ x \geq 132.17 \]
\[ x \leq +283.41 \]
are both basic lenient inequalities.

5. A basic problem with thus be a problem in which
• the data set consists of number-phrases
• the formula is a basic formula
• the common denominator has been factored out and declared up-front.

EXAMPLE 30. Given the basic problem in Dollars where
- The data set is:
  \[ \{2, 3, 4, 5, 6, 7, 8\} \]
- The formula is:
  \[ x > 5 \]
  (where \( x \) is an unspecified numerator and 5 is the gauge numerator)
the solution subset is
  \[ \{6, 7, 8\} \]

7.4 Basic Problems

Given a basic problem involving counting number-phrases,

i. We determine the solution subset by replacing the unspecified numerator successively by each and every numerator in the data set. We then have comparison sentences that are true or false depending on
• which one of the six verbs is the verb in the formula.
• which way, up or down or not at all, we have to count from the numerator replacing the unspecified numerator to the given gauge numerator
(See Chapter 2.)

ii. We represent the solution subset:
• To graph the solution subset, we will use:
  - a solid dot to represent a solution: \( \bullet \)
  - a hollow dot to represent a non-solution: \( \circ \)
• To name the solution subset, we will use, just as for data sets, two curly brackets, \( \{ \quad \} \), and write the solutions in-between the curly brackets.

1. Usually, a problem has both non-solutions and solutions.

EXAMPLE 31.

i. In the real world, Jack has the following collection of one-dollar bills
empty

So the bids that he can at all make in an auction (set of selectable collections) are:

If the starting bid for a particular object is three dollars (a selectable collection), then the bids that he could make (select subset) would be:

II. On paper, we represent this by the following problem:
- We represent the set of selectable collections by the data set:
  \[ \{1, 2, 3, 4, 5\} \text{ Dollars} \]
- We represent the requirement that the bid must be no less than three dollars by the formula
  \[ x \geq 3 \]

III. To determine the solution subset we check each and every numerator in the data set. The verb \( \geq \) requires that, from the numerator that replaces the unspecified numerator to the gauge numerator, we must count down or must not count.

- \( x \geq 3 \)
  - \( x \geq 3 \mid x = 1 \): false because, from 1 to 3, we must count up
  - \( x \geq 3 \mid x = 2 \): false because, from 2 to 3, we must count up
  - \( x \geq 3 \mid x = 3 \): true because, from 3 to 3, we must not count
  - \( x \geq 3 \mid x = 4 \): true because, from 4 to 3, we must count down
  - \( x \geq 3 \mid x = 5 \): true because, from 5 to 3, we must count down

So:

- 1 is a non-solution
- 2 is a non-solution
- 3 is a solution
- 4 is a solution
- 5 is a solution

IV. We represent the solution subset

- The graph of the solution subset is:

- The name of the solution subset is:
  \[ \{3, 4, 5\} \text{ Dollars} \]

2. Occasionally, it can happen that there is no solution in which case we say that the solution subset is empty.

**Example 32.**

I. In the real world, Jack has the following collection of one-dollar bills
So the bids that he can at all make in an auction (set of selectable collections) are:

If the starting bid for a particular object is seven dollars (a selectable collection), then he would not be able to make any bid (the select subset is empty):

II. On paper, we represent this by the following problem:

- We represent the set of selectable collections by the data set:
  \{1, 2, 3, 4, 5\} Dollars

- We represent the requirement that the bid must be no less than three dollars by the formula
  \( x \geq 7 \)

III. To determine the solution subset we check each and every numerator in the data set. The verb \( \geq \) requires that, from the numerator that replaces the unspecified numerator to the gauge numerator, we must count down or must not count.

- \( x \geq 7 \big| x = 1 \) is FALSE because, from 1 to 7, we must count up
- \( x \geq 7 \big| x = 2 \) is FALSE because, from 2 to 7, we must count up
- \( x \geq 7 \big| x = 3 \) is FALSE because, from 3 to 7, we must count up
- \( x \geq 7 \big| x = 4 \) is FALSE because, from 4 to 7, we must count up
- \( x \geq 7 \big| x = 5 \) is FALSE because, from 5 to 7, we must count up

So:

- 1 is a non-solution
- 2 is a non-solution
- 3 is a non-solution
- 4 is a non-solution
- 5 is a non-solution

IV. We represent the solution subset

- The graph of the solution subset is:

- The name of the solution subset is:
  \{ \} Dollars

3. Occasionally, it can happen that there is no non-solution in which case we say that the solution subset is full.

**Example 33.**

I. In the real world, Jack has the following collection of one-dollar bills
So the bids that he can at all make in an auction (set of selectable collections) are:

If the starting bid for a particular object is one dollars (a selectable collection), then he can any bid any selectable collection (the select subset is full):

II. On paper, we represent this by the following problem:

- We represent the set of selectable collections by the data set:
  \{1, 2, 3, 4, 5\} Dollars
- We represent the requirement that the bid must be no less than three dollars by the formula
  \( x \geq 1 \)

III. To determine the solution subset we check each and every numerator in the data set. The verb \( \geq \) requires that, from the numerator that replaces the unspecified numerator to the gauge numerator, we must count down or must not count.

- \( x \geq 1 \mid_{x=1} \) is TRUE because, from 1 to 1, we must not count
- \( x \geq 1 \mid_{x=2} \) is TRUE because, from 2 to 1, we must count down
- \( x \geq 1 \mid_{x=3} \) is TRUE because, from 3 to 1, we must count down
- \( x \geq 1 \mid_{x=4} \) is TRUE because, from 4 to 1, we must count down
- \( x \geq 1 \mid_{x=5} \) is TRUE because, from 5 to 1, we must count down

So:

- 1 is a solution
- 2 is a solution
- 3 is a solution
- 4 is a solution
- 5 is a solution

IV. We represent the solution subset

- The graph of the solution subset is:

- The name of the solution subset is:
  \{1, 2, 3, 4, 5\} Dollars

4. When the data set is infinite, we cannot check every numerator in the data set and we must make the case that beyond a certain numerator, the numerators are all solutions or all non-solutions.

**EXAMPLE 34.**

I. On paper, we represent such a situation by the following problem:

- We represent the set of selectable collections by the data set:
  \{1, 2, 3, 4, 5, \ldots\} Dollars

where ... is read “and so on”.
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- We represent the requirement that the bid must be no less than three dollars by the formula
  \[ x \geq 3 \]

II. To determine the solution subset we are supposed to check each and every numerator in the data set. The verb \( \geq \) requires that, from the numerator that replaces the unspecified numerator to the gauge numerator, we must count down or must not count.

i. We start by checking each and every numerator in the data set until we pass the gauge numerator 3:
   - \( x \geq 3 \mid x=1 \) is FALSE because, from 1 to 3, we must count up
   - \( x \geq 3 \mid x=2 \) is FALSE because, from 2 to 3, we must count up
   - \( x \geq 3 \mid x=3 \) is TRUE because, from 3 to 3, we must not count
   - \( x \geq 3 \mid x=4 \) is TRUE because, from 4 to 3, we must count down

So:

1 is a non-solution
2 is a non-solution
3 is a solution
4 is a solution

ii. We now make the case that any numerator beyond 4, that is 5, 6, 7, …, is a solution:
   - Since, from any numerator beyond 4, that is 5, 6, 7, …, to 4, we must count down,
   - And since, from 4 to the gauge 3, we must count down,
   - It follows that from any numerator beyond 4, that is 5, 6, 7, …, to the gauge 3, we must count down.

So, any numerator beyond 4, that is 5, 6, 7, … is also going to be a solution.

III. We represent the solution subset
- The graph of the solution subset is:

\[ \begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 1 & 2 & 3 & 4 \\
\end{array} \]

\[ \xrightarrow{\text{and so on}} \text{ Dollars} \]

where we actually write “and so on” because … would run the risk of not being seen.
- The name of the solution subset is:
  \[ \{1, 2, 3, 4, 5, \ldots\} \text{ Dollars} \]
  where we use … to mean “and so on”.

5. When the data set involves signed numerators, we proceed essentially in the same manner as with plain numerators.

EXAMPLE 35.

I. On paper, we represent such a situation by the following problem:
- We represent the set of selectable collections by the data set:
  \[ \{ -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, \ldots \} \text{ Dollars} \]
  where … is read “and so on”.

#
• We represent the requirement that the balance must be more than a three dollar debt by the formula
  \[ x > -3 \]

II. i. We start by checking each and every numerator in the data set until we pass the gauge numerator 3:
  \[ x \geq -3 \mid x = -5 \] is false because, from -5 to -3, we must count up
  \[ x \geq -3 \mid x = -4 \] is false because, from -4 to -3, we must count up
  \[ x \geq -3 \mid x = -3 \] is true because, from -3 to -3, we must not count
  \[ x \geq -3 \mid x = -2 \] is true because, from -2 to -3, we must count down

So:

-5 is a non-solution
-4 is a non-solution
-3 is a solution
-2 is a solution

ii. We now make the case that any numerator beyond -2, that is -1, 0, +1, +2, ... is a solution:
  • Since, from any numerator beyond -2, that is -1, 0, +1, +2, ..., to -2, we must count down,
  • And since, from -2 to the gauge -3, we must count down,
  • It follows that from any numerator beyond -2, that is -1, 0, +1, +2, ... to the gauge -3, we must count down.

So, any numerator beyond -2, that is -1, 0, +1, +2, ..., is also going to be a solution.

III. We represent the solution subset
  • The graph of the solution subset is:

    ![Graph representation]

    where we actually write “and so on” because ... would run the risk of not being seen.
  • The name of the solution subset is:

    \( \{ -3, -2, -1, 0, +1, +2, +3, \ldots \} \) Dollars

    where we use ... to mean "and so on."