repeat
involution

Chapter 13

Repeated Multiplications and Divisions

Given a *number-phrase* we investigate what is involved in **repeated** multiplications or *repeated* divisions by a given *numerator*, something which used to be called **involution**[1].

13.1 A Problem With English

English can be confusing when we want to *indicate* "how many times" an operation is to be repeated.

1. One source of confusion is the word "times" because *multiplication* may not be involved at all.

EXAMPLE 1. When we tell someone

Divide 375 **Dollars** 3 times by 5

multiplication is not involved and we just mean:

Divide 375 **Dollars**

i. a first time by 5—which gives 75 **Dollars** as a result,
ii. a second time by 5—which gives 15 **Dollars** as a result,
iii. a third time by 5—which gives 3 **Dollars** as a result.

NOTE. In fact, the use of "first time", "second time", etc is also a bit misleading since, when we "divide for the second time", we are not dividing the *initial number-phrase* a second time but *the result of the first division* for the first time. Etc.

[1]Educologists will surely deplore this departure from the usual "modern" treatment. Yet, it is difficult to see how conflating *unary operators* and *binary operations* can be helpful.

coefficient
base
plain exponent

2. Another source of confusion is when we do not pay attention to the exact place of the word "by".

EXAMPLE 2. While, as we saw in **EXAMPLE** 1,

Divide 375 **Dollars** 3 times *by* 5

results in

3 **Dollars**

it is easily confused with

Divide 375 **Dollars** *by* 3 times 5

that is

Divide 375 **Dollars** *by* 15

whose results is

25 **Dollars**

3. A workaround would seem just to avoid using the word "by" but it is awkward and even misleading when we *say* it and downright dangerous when we *write* it.

EXAMPLE 3. To *say*

multiply 7 **Dollars** by 2, 3 times

can be correctly understood but requires to stop markedly after the 2 as, otherwise, it will be understood to mean

multiply 7 **Dollars** by 2 OR by 3.

EXAMPLE 4. To *write*

multiply 7 **Dollars** by 2, 3 times

can be correctly understood but requires paying attention to the comma between the 2 and the 3 as otherwise it will be understood to mean

Multiply 7 **Dollars** by 23

4. What we will now do will be to develop a *specialized language* to deal with repeated operations. Perhaps surprisingly, though, writing specifying-phrases for *repeated* operations is not quite a simple matter.

13.2 Templates

We begin by looking at the way we actually go about repeating operations.

1. Given a *number-phrase*, whose *numerator* we will refer to as the **coefficient**, and:

- given a *numerator*, called the **base**, by which the given number-phrase is to be repeatedly multiplied or repeatedly divided,
- given a *numerator*, called the **plain exponent**, to indicate how many multiplications or how many divisions we want done on the *coefficient*,

the simplest way to *specify* how many repeated multiplications or how many divisions we want done on the *number-coefficient* is to use a **staggered**

template in which each operation is done on a separate line with a separate **copy** of the base.

staggered template
copy

EXAMPLE 5. When we want the number-phrase $+7$ **Dollars** *multiplied* by 6 copies of -2, we say that

- the *coefficient* is $+7$,
- the *base* from which we make the *copies* is -2,
- the *plain exponent* is 6

and we write the following *staggered template*:

$+7$ **Dollars** $\otimes -2$ (1^{st} multiplication by -2)
$\boxed{} \otimes -2$ (2^{nd} multiplication by -2)
$\boxed{} \otimes -2$ (3^{rd} multiplication by -2)
$\boxed{} \otimes -2$ (4^{th} multiplication by -2)
$\boxed{} \otimes -2$ (5^{th} multiplication by -2)
$\boxed{} \otimes -2$ (6^{th} multiplication by -2)
$\boxed{}$ (Result of the repeated multiplications)

The staggered template specifies what is to be done at each stage and therefore what the result will be:

$+7$ **Dollars** $\otimes -2$ (1^{st} multiplication by -2)
$\boxed{-14 \text{ Dollars}} \otimes -2$ (2^{nd} multiplication by -2)
$\boxed{+28 \text{ Dollars}} \otimes -2$ (3^{rd} multiplication by -2)
$\boxed{-56 \text{ Dollars}} \otimes -2$ (4^{th} multiplication by -2)
$\boxed{+112 \text{ Dollars}} \otimes -2$ (5^{th} multiplication by -2)
$\boxed{-224 \text{ Dollars}} \otimes -2$ (6^{th} multiplication by -2)
$\boxed{+448 \text{ Dollars}}$ (Result of the repeated multiplications)

EXAMPLE 6. When we want the number-phrase $+112$ **Dollars** *divided* by 4 copies of -2, we say that

- the *coefficient* is $+112$,
- the *base* from which we make the *copies* is -2,
- the *plain exponent* is 4

and we write the following *staggered template*:

Divide $+112$ **Dollars** by 4 *copies* of -2

we use the *staggered template*:

$\underbrace{+112 \text{ Dollars} \oplus\!\!\!\!\div -2}$ (1st division by -2)

$\underbrace{\boxed{\qquad\quad} \div -2}$ (2nd division by -2)

$\underbrace{\boxed{\qquad\quad} \div -2}$ (3rd division by -2)

$\underbrace{\boxed{\qquad\quad} \div -2}$ (4th division by -2)

$\boxed{\qquad\quad}$ (Result of the repeated divisions)

The staggered template specifies what is to be done at each stage and therefore what the result will be:

$\underbrace{+112 \text{ Dollars} \div -2}$ (1st division by -2)

$\underbrace{\boxed{-56 \text{ Dollars}} \div -2}$ (2nd division by -2)

$\underbrace{\boxed{+28 \text{ Dollars}} \div -2}$ (3rd division by -2)

$\underbrace{\boxed{-14 \text{ Dollars}} \div -2}$ (4th division by -2)

$\boxed{+7 \text{ Dollars}}$ (Result of the repeated divisions)

2. As usual, instead of writing the denominator on each line, we can *declare* the denominator up front and then write the staggered template just for the *numerators*.

EXAMPLE 7. When we want the number-phrase $+7$ **Dollars** *multiplied* by 6 copies of -2, we can

i. *declare* that the template is in **Dollars**

ii. write the staggered template just for the *numerators*

$\underbrace{+7 \otimes -2}$ (1st multiplication by -2)

$\underbrace{\boxed{\quad} \otimes -2}$ (2nd multiplication by -2)

$\underbrace{\boxed{\quad} \otimes -2}$ (3rd multiplication by -2)

$\underbrace{\boxed{\quad} \otimes -2}$ (4th multiplication by -2)

$\underbrace{\boxed{\quad} \otimes -2}$ (5th multiplication by -2)

$\underbrace{\boxed{\quad} \otimes -2}$ (6th multiplication by -2)

$\boxed{\quad}$ (Result of the repeated multiplications)

The staggered template specifies what the *numerator* of the result will be and the declaration specifies that the *denominator* is **Dollars**.

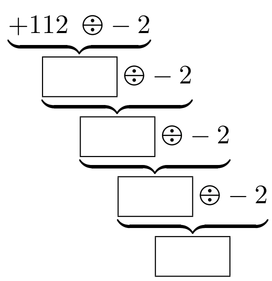
EXAMPLE 8. When we want the number-phrase $+112$ **Dollars** *divided* by 4 copies of -2, we say that

- the *coefficient* is $+112$,

in-line template

- the *base* from which we make the *copies* is -2,
- the *plain exponent* is 4

and we write the following *staggered template* in **Dollars**:



(1^{st} division by -2)

(2^{nd} division by -2)

(3^{rd} division by -2)

(4^{th} division by -2)

(Result of the repeated divisions)

The staggered template specifies what is to be done at each stage and therefore what the numerator of the result in **Dollars** will be.

3. Quite often, though, we will not want to *get* the actual result but just be able to *discuss* the repeated operations and, in that case, the use of *staggered* templates is cumbersome. So, what we will do is to let the boxes "go without saying" which will allow us to write an **in-line template**, that is:

i. For the *numerators*, we write on a single line:

i. The *coefficient*,

ii. The *operation symbol* followed by the 1^{st} *copy* of the *base*

iii. The *operation symbol* followed by the 2^{nd} *copy* of the *base*

iv. The *operation symbol* followed by the 3^{rd} *copy* of the *base*

v. Etc until all *copies* specified by the *plain exponent* have been written.

ii. For the *denominator*, we have a choice:

– We can *declare* the *denominator* up front and then write the in-line template for the *numerators*,

– We can write the in-line template for the *numerators within square brackets* and then write the *denominator*.

EXAMPLE 9. Instead of writing the *staggered* template in **Dollars**

$\underbrace{+7 \otimes -2}$	(1^{st} multiplication by -2)
$\underbrace{\square \otimes -2}$	(2^{nd} multiplication by -2)
$\underbrace{\square \otimes -2}$	(3^{rd} multiplication by -2)
$\underbrace{\square \otimes -2}$	(4^{th} multiplication by -2)
$\underbrace{\square \otimes -2}$	(5^{th} multiplication by -2)
$\underbrace{\square \otimes -2}$	(6^{th} multiplication by -2)
$\square$	(Result of the repeated multiplications)

we can:

- *Declare up front* that the *in-line* template is in **Dollars** and then write:

$$+17 \otimes -2 \ \otimes -2 \ \otimes -2 \ \otimes -2 \ \otimes -2 \ \otimes -2$$

or

- Write the in-line template for the numerators *within square brackets* and then write the denominator **Dollars**

$$\left[+17 \otimes -2 \ \otimes -2 \ \otimes -2 \ \otimes -2 \ \otimes -2 \ \otimes -2\right] \textbf{Dollars}$$

EXAMPLE 10. Instead of writing the *staggered* template in **Dollars**

$\underbrace{-208 \oplus - 2}$	(1^{st} division by -2)
$\underbrace{\square \oplus - 2}$	(2^{nd} division by -2)
$\underbrace{\square \oplus - 2}$	(3^{rd} division by -2)
$\underbrace{\square \oplus - 2}$	(4^{th} division by -2)
$\square$	(Result of the repeated divisions)

we can

- *Declare up front* that the *in-line* template is in **Dollars** and then write:

$$-208 \oplus - 2 \ \oplus - 2 \ \oplus - 2 \ \oplus - 2$$

- Write the in-line template for the numerators *within square brackets* and then write the denominator **Dollars**

$$\left[-208 \oplus - 2 \ \oplus - 2 \ \oplus - 2 \ \oplus - 2\right] \textbf{Dollars}$$

13.3 The Order of Operations

The use of *in-line* templates for repeated operations, though, poses a problem: how do we know for sure in what order the recipient of an *in-line* template is going to do the operations?

The reason this can be a problem is that this order can make all the difference between the recipient arriving at the intended result and the recipient arriving at something completely irrelevant.

1. When the operation being repeated is *multiplication*, it turns out that the order in which the operations are done does *not* matter

EXAMPLE 11. Given the in-line template in **Dollars**

$$17 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

the recipient might choose to compute it as

$$\underbrace{\underbrace{\underbrace{\underbrace{\underbrace{\underbrace{17\times 2}_{34}\times 2}_{68}\times 2}_{136}\times 2}_{272}\times 2}_{544}\times 2}_{1088}$$

or the recipient might choose to compute it as

$$\underbrace{17\times\underbrace{2\times\underbrace{2\times\underbrace{2\times\underbrace{2\times\underbrace{2\times 2}_{4}}_{8}}_{16}}_{32}}_{64}}_{1088}$$

or as

$$\underbrace{17\times\underbrace{2\times\underbrace{2\times\underbrace{\underbrace{\underbrace{2\times 2}_{4}\times 2}_{8}\times 2}_{16}}_{32}}_{64}}_{1088}$$

etc but, it does not matter as the result will always be 1088.

However, proving *in general* that the *order* in which the *multiplications* are done does *not* matter takes some work because, as the number of copies gets large, the number of ways in which the multiplications could be done gets even larger and yet, to be able to make a *general statement*, we would have to make sure that *all* of these ways have been accounted for. So, for the sake of time, in the case of repeated *multiplications*, we will take the following for granted:

THEOREM 6. *The order in which* multiplications *are done does* not *matter.*

2. In the case of repeated *division*, though, the order usually makes a *huge* difference.

EXAMPLE 12. Given the in-line template in **Dollars**

$$448 \div 2 \div 2 \div 2 \div 2 \div 2 \div 2$$

and while the recipient might indeed choose to compute is as

$$\underbrace{\underbrace{\underbrace{\underbrace{\underbrace{\underbrace{448 \div 2}_{224} \div 2}_{112} \div 2}_{56} \div 2}_{28} \div 2}_{14} \div 2}_{7}$$

the recipient might also choose to compute it as

$$\underbrace{448 \div \underbrace{2 \div \underbrace{2 \div \underbrace{2 \div \underbrace{2 \div \underbrace{2 \div 2}_{1}}_{2}}_{1}}_{2}}_{1}}_{448}$$

or as

$$\underbrace{448 \div \underbrace{2 \div \underbrace{2 \div \underbrace{\underbrace{\underbrace{2 \div 2}_{1} \div 2}_{0.5} \div 2}_{0.25}}_{8}}_{0.25}}_{1796}$$

etc

Thus, in the case of repeated *divisions* it is crucial to agree on the order in which to do them and so, in the absence of any instructions to that effect, we will use

DEFAULT RULE # 5. *The order in which* divisions *are to be done is* from left to right.

13.4 The Way to Powers

Eventually, we will devise a very powerful language to deal both with repeated multiplications and repeated divisions but, before we can do that, we need to clear the way.

1. While, as we have seen, 1 does tend to "go without saying", what we can do when the *coefficient* in a repeated operation is 1 depends on whether the operation being repeated is *multiplication* or *division*.

a. When it is *multiplication* that is being repeated, we can let the coefficient 1 go without saying. However, the number of multiplications is then one less than the number of copies[2].

EXAMPLE 13. Given the in-line template in **Dollars**

$$1 \times 3 \times 3 \times 3 \times 3 \times 3$$

we *can* write instead

$$3 \times 3 \times 3 \times 3 \times 3$$

because we get 243 either way.
However, while we still have five copies of 3, we now have only four multiplications.

b. When it is *division* that is being repeated, we *must* write the coefficient 1 as, if we did not, we would be getting a different result.

EXAMPLE 14. Given the in-line template in **Dollars**

$$1 \div 2 \div 2 \div 2 \div 2 \div 2$$

the 1 cannot go without saying because, while the *given* in-line template computes to $\frac{1}{32}$, if we don't write the coefficient 1, we get an in-line template with coefficient 2 to be divided by four copies of 2:

$$2 \div 2 \div 2 \div 2 \div 2$$

which computes to $\frac{1}{8}$.

2. *Repeated divisions* are related to *repeated multiplications*. Indeed,

- instead of dividing a coefficient by a number of copies of the *base*,
- we can[3]:
  i. multiply 1 repeatedly by the number of copies of the base,
  ii. divide the coefficient by the *result* of the repeated multiplication.

EXAMPLE 15. Given the in-line template in **Dollars**

$$448 \div 2 \div 2 \div 2 \div 2 \div 2 \div 2$$

[2]Educologists will correctly point out that while 1× can "go without saying", this is really where multiplication as a *binary* operation comes in.

[3]Educologist will point out that, essentially, this is just the fact that, instead of dividing by a numerator, we can multiply by its *reciprocal*.

instead of computing it as follows:

$$\underbrace{\underbrace{\underbrace{\underbrace{\underbrace{\underbrace{448 \div 2}_{224} \div 2}_{112} \div 2}_{56} \div 2}_{28} \div 2}_{14} \div 2}_{\boxed{7}}$$

we can proceed as follows:

i. We *multiply* 1 by the 6 copies of 2

$$\underbrace{\underbrace{\underbrace{\underbrace{\underbrace{\underbrace{1 \times 2}_{2} \times 2}_{4} \times 2}_{8} \times 2}_{16} \times 2}_{32} \times 2}_{64}$$

ii. We *divide* the coefficient 448 by the *result* of this repeated multiplications:

$$448 \div 64 = \boxed{7}$$

which indeed gives us the same result as the repeated division.

The advantage of this second way of computing in-line templates involving repeated *divisions* is that while we now have one more *operation* than we had *divisions*, the first multiplication, multiplying the coefficient 1 by the first copy of the base, is no work and, as we saw above, need in fact not even be written so that the number of operations *requiring work* is the same in both cases. But now all operations except one are *multiplications* which are a lot less work than *divisions*.

However, here again, proving *in general* that the results are always the same takes some work so that, for the sake of saving time, we will take for granted that:

THEOREM 7. *A repeated division is the same as a single division of the coefficient by the result of* 1 *multiplied repeatedly by the same number of copies of the base.*

$$\boxed{\text{Coefficient} \;\text{⨸}\; \text{copies} = \text{Coefficient} \;\text{⨸}\; [1 \;\otimes\; \text{copies}]}$$

3. In order to *specify* the second way of computing, we can write either:

bracket in-line template
fraction-like template
fraction bar

- A **bracket in-line template** where we write:
    i. The coefficient followed by a division symbol,
    ii. A pair of square brackets within which we write
    iii. 1 repeatedly multiplied by the same number of copies of the base.

EXAMPLE 16. Instead of writing the in-line template in **Dollars** as

$$+448 \odiv -2 \odiv -2 \odiv -2 \odiv -2 \odiv -2 \odiv -2$$

we can write the *bracket in-line template* in **Dollars** as

$$+448 \div \left[+1 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2\right]$$

or as

$$+448 \div \left[-2 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2\right]$$

or

- A **fraction-like template** where we write:
    i. The coefficient and, underneath,
    ii. A **fraction bar** and, underneath
    iii. 1 repeatedly multiplied by the same number of copies of the base

with the 1 able to "go without saying".
underneath and the repeated multiplication underneath the *bar*,

EXAMPLE 17. Instead of writing the in-line template in **Dollars**

$$+448 \div -2 \div -2 \div -2 \div -2 \div -2 \div -2$$

we can write the in-line template in **Dollars** as

$$\frac{+448}{+1 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2}$$

or as

$$\frac{+448}{-2 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2}$$

NOTE. Whether we use a *bracket in-line template* or a *fraction-like template*, we need not write the 1 as, either way, there is something to remind us that the multiplications have to be done *first*:

- The *square brackets*

or

- The *fraction bar*

In general, though, we will prefer to use *fraction-like* templates with the 1 "going without saying".

In other words, instead of:

$$\boxed{\text{Coefficient } \odiv \text{ copies} = \text{Coefficient} \odiv [1 \otimes \text{ copies}]}$$

we prefer to write

monomial specifying-phrase
separator
signed exponent
superscript
signed power

$$\boxed{\text{Coefficient } \oplus\!\!\!\!\div \text{ copies} = \frac{\text{Coefficient}}{\text{copies}}}$$

but, even though both sides are *read* as

"Coefficient divided by copies"

- the *division symbol* ⨸ on the left side of =

$$\boxed{\text{Coefficient } \oplus\!\!\!\!\div \text{ copies} = }$$

  says that the coefficient is to be divided *repeatedly* by the copies of the base
- the *fraction bar* on the right side of =

$$\boxed{= \frac{\text{Coefficient}}{\text{copies}}}$$

  says that the coefficient is to be divided by the *result* of the multiplication of 1 by the copies of the base.

13.5 Power Language

We are now ready to introduce a way of writing specifying-phrases that will work both for *repeated multiplications* and for *repeated divisions*.

1. The idea is to *write* just the *coefficient*, the *base*, the *number* of copies and whether the coefficient should be *multiplied* or *divided* by the copies. More precisely, in order to write a new kind of specifying-phrase which we will call a **monomial specifying-phrase**,

i. We write its *numerator*, that is we write:

i. The *coefficient*,

ii. The *multiplication* symbol $\times$ or $\otimes$ (depending on whether the numerators are *plain* or *signed*) as **separator** followed by the *base*,

iii. A **signed exponent**, that is a signed numerator

- whose *sign* is
  - $+$ when the coefficient is to be *multiplied* by the copies
  - $-$ when the coefficient is to be *divided* by the copies
- whose *size* is the number of copies

In order to be *separated* from the *base*, the *signed exponent* must be written as a **superscript**, that is small and raised a bit above the base line.

ii. We write its *denominator* if it has not been *declared* up front.

The *base* together with the *signed-exponent* is called a **signed power**.

We then *read* monomial specifying-phrases as

"Coefficient *multiplied/divided* by number of *copies* of the base"

EXAMPLE 18. Given the in-line *template* in **Dollars**

$$17 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

- In order to *write* the monomial specifying-phrase,
  i. We write the *coefficient* 17:
  $$17$$
  ii. We write the *multiplication* symbol $\times$ as *separator* followed by the *base* 2:
  $$17 \times 2$$
  iii. We write the *signed exponent* as a *superscript* with $+$ to indicate that the coefficient is to be *multiplied* by the 6 copies of the base 2:
  $$17 \times 2^{+6}$$
- We *read* the monomial specifying-phrase
  $$\boxed{17 \times 2^{+6}}$$
  as
  $$\boxed{\text{17 multiplied by 6 copies of 2}}$$

EXAMPLE 19. Given the in-line *template*

$$448 \div \left[2 \times 2 \times 2 \times 2 \times 2 \times 2\right]$$

- In order to *write* the monomial specifying-phrase,
  i. We write the *coefficient* 448:
  $$448$$
  ii. We write the *multiplication* symbol $\times$ as *separator* followed by the *base* 2:
  $$448 \times 2$$
  iii. We write the *signed exponent* with $-$ to indicate that the coefficient is to be *divided* by 6 copies of the base 2:
  $$448 \times 2^{-6}$$
- We *read* the monomial specifying-phrase
  $$\boxed{448 \times 2^{-6}}$$
  as
  $$\boxed{\text{448 divided by 6 copies of 2}}$$

NOTE. In other words, here, $\times$ is really only a *separator* and has nothing to do with the kind of *repeated operation* we are specifying. While this way of writing things might seem rather strange, we will see in the next section how it turns out to make excellent sense.

2. As it happens, though, there is *no* procedure for identifying *monomial specifying-phrases* other than the procedures corresponding to *staggered templates*.

EXAMPLE 20. Given the following monomial specifying-phrase in **Dollars**

$$17 \times 2^{+6}$$

there is *no* way to identify it other than doing

$$\underbrace{17 \times 2}$$
$$\underbrace{34 \times 2}$$

Laurent monomial specifying-phrase
plain monomial specifying-phrase

$$\underbrace{\underbrace{\underbrace{\underbrace{68 \times 2}_{136} \times 2}_{272} \times 2}_{544} \times 2}_{1088}$$

This is in sharp contrast with the case of *repeated additions* for which there is a much shorter procedure for getting the result of repeated additions that is based on *multiplication* and with the case of *repeated subtractions* for which there is a much shorter procedure for getting the result based on *division*.

3. It is customary to distinguish monomial specifying-phrases in which the exponent has to be positive or 0 from monomial specifying-phrases in which the exponent can have any sign.

We will use the following names:

- A **Laurent monomial specifying-phrase** is a monomial specifying-phrases in which the exponent is a numerator that can have any sign.
- A **plain monomial specifying-phrase** is a monomial specifying-phrases in which the exponent is a numerator that can be only positive or 0 or, in other words, that can only be a plain numerator.