Chapter 13

Repeated Multiplications and Divisions

Given a number-phrase we investigate what is involved in repeated multiplications or repeated divisions by a given numerator, something which used to be called involution\(^1\).

13.1 A Problem With English

English can be confusing when we want to indicate “how many times” an operation is to be repeated.

1. One source of confusion is the word “times” because multiplication may not be involved at all.

**Example 1.** When we tell someone

\[
\text{Divide 375 Dollars 3 times by 5}
\]

multiplication is not involved and we just mean:

\[
\text{Divide 375 Dollars}
\]

i. a first time by 5—which gives 75 Dollars as a result,

ii. a second time by 5—which gives 15 Dollars as a result,

iii. a third time by 5—which gives 3 Dollars as a result.

**Note.** In fact, the use of “first time”, “second time”, etc is also a bit misleading since, when we “divide for the second time”, we are not dividing the initial number-phrase a second time but the result of the first division for the first time. Etc.

\(^1\)Educologists will surely deplore this departure from the usual “modern” treatment. Yet, it is difficult to see how conflating unary operators and binary operations can be helpful.
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2. Another source of confusion is when we do not pay attention to the exact place of the word “by”.

**Example 2.** While, as we saw in **Example 1,**

\[
\text{Divide } 375 \text{ Dollars } 3 \text{ times by } 5
\]

results in

3 Dollars

it is easily confused with

\[
\text{Divide } 375 \text{ Dollars by } 3 \text{ times } 5
\]

that is

\[
\text{Divide } 375 \text{ Dollars by } 15
\]

whose results is

25 Dollars

3. A workaround would seem just to avoid using the word “by” but it is awkward and even misleading when we say it and downright dangerous when we write it.

**Example 3.** To say

\[
\text{multiply } 7 \text{ Dollars by } 2, 3 \text{ times}
\]

can be correctly understood but requires to stop markedly after the 2 as, otherwise, it will be understood to mean

\[
\text{multiply } 7 \text{ Dollars by } 2 \text{ OR by } 3.
\]

**Example 4.** To write

\[
\text{multiply } 7 \text{ Dollars by } 2, 3 \text{ times}
\]

can be correctly understood but requires paying attention to the comma between the 2 and the 3 as otherwise it will be understood to mean

\[
\text{Multiply } 7 \text{ Dollars by 23}
\]

4. What we will now do will be to develop a *specialized language* to deal with repeated operations. Perhaps surprisingly, though, writing specifying-phrases for repeated operations is not quite a simple matter.

### 13.2 Templates

We begin by looking at the way we actually go about repeating operations.

1. Given a *number-phrase*, whose *numerator* we will refer to as the *coefficient*, and:
   - given a *numerator*, called the *base*, by which the given number-phrase is to be repeatedly multiplied or repeatedly divided,
   - given a *numerator*, called the *plain exponent*, to indicate how many multiplications or how many divisions we want done on the *coefficient*,

the simplest way to *specify* how many repeated multiplications or how many divisions we want done on the *number-coefficient* is to use a *staggered*
template in which each operation is done on a separate line with a separate copy of the base.

**Example 5.** When we want the number-phrase +7 Dollars multiplied by 6 copies of −2, we say that

- the coefficient is +7,
- the base from which we make the copies is −2,
- the plain exponent is 6

and we write the following staggered template:

\[
\begin{array}{c}
+7 \text{ Dollars} \otimes -2 \\
\otimes -2 \\
\otimes -2 \\
\otimes -2 \\
\otimes -2 \\
\otimes -2 \\
\end{array}
\]

(1st multiplication by −2)

(2nd multiplication by −2)

(3rd multiplication by −2)

(4th multiplication by −2)

(5th multiplication by −2)

(6th multiplication by −2)

(Result of the repeated multiplications)

The staggered template specifies what is to be done at each stage and therefore what the result will be:

\[
\begin{array}{c}
+7 \text{ Dollars} \otimes -2 \\
-14 \text{ Dollars} \otimes -2 \\
+28 \text{ Dollars} \otimes -2 \\
-56 \text{ Dollars} \otimes -2 \\
+112 \text{ Dollars} \otimes -2 \\
-224 \text{ Dollars} \otimes -2 \\
+448 \text{ Dollars} \\
\end{array}
\]

(1st multiplication by −2)

(2nd multiplication by −2)

(3rd multiplication by −2)

(4th multiplication by −2)

(5th multiplication by −2)

(6th multiplication by −2)

(Result of the repeated multiplications)

**Example 6.** When we want the number-phrase +112 Dollars divided by 4 copies of −2, we say that

- the coefficient is +112,
- the base from which we make the copies is −2,
- the plain exponent is 4

and we write the following staggered template:

\[
\text{Divide } +112 \text{ Dollars by 4 copies of } -2
\]

we use the staggered template:
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\[
\begin{align*}
\text{\textbf{+112 Dollars}} & \otimes -2 \quad (1^{\text{st}} \text{ division by } -2) \\
\text{\textbf{-56 Dollars}} & \otimes -2 \quad (2^{\text{nd}} \text{ division by } -2) \\
\text{\textbf{+28 Dollars}} & \otimes -2 \quad (3^{\text{rd}} \text{ division by } -2) \\
\text{\textbf{-14 Dollars}} & \otimes -2 \quad (4^{\text{th}} \text{ division by } -2) \\
\text{\textbf{+7 Dollars}} & \quad \text{(Result of the repeated divisions)}
\end{align*}
\]

The staggered template specifies what is to be done at each stage and therefore what the result will be:

\[
\begin{align*}
\text{\textbf{+112 Dollars}} & \otimes -2 \quad (1^{\text{st}} \text{ division by } -2) \\
\text{\textbf{-56 Dollars}} & \otimes -2 \quad (2^{\text{nd}} \text{ division by } -2) \\
\text{\textbf{+28 Dollars}} & \otimes -2 \quad (3^{\text{rd}} \text{ division by } -2) \\
\text{\textbf{-14 Dollars}} & \otimes -2 \quad (4^{\text{th}} \text{ division by } -2) \\
\text{\textbf{+7 Dollars}} & \quad \text{(Result of the repeated divisions)}
\end{align*}
\]

2. As usual, instead of writing the denominator on each line, we can declare the denominator up front and then write the staggered template just for the numerators.

**Example 7.** When we want the number-phrase \textbf{+7 Dollars multiplied by} 6 copies of \(-2\), we can

i. declare that the template is in \textbf{Dollars}

ii. write the staggered template just for the numerators

\[
\begin{align*}
\text{\textbf{+7 \otimes -2}} & \quad (1^{\text{st}} \text{ multiplication by } -2) \\
\text{\textbf{\otimes -2}} & \quad (2^{\text{nd}} \text{ multiplication by } -2) \\
\text{\textbf{\otimes -2}} & \quad (3^{\text{rd}} \text{ multiplication by } -2) \\
\text{\textbf{\otimes -2}} & \quad (4^{\text{th}} \text{ multiplication by } -2) \\
\text{\textbf{\otimes -2}} & \quad (5^{\text{th}} \text{ multiplication by } -2) \\
\text{\textbf{\otimes -2}} & \quad (6^{\text{th}} \text{ multiplication by } -2) \\
\text{\textbf{\quad (Result of the repeated multiplications)}}
\end{align*}
\]

The staggered template specifies what the numerator of the result will be and the declaration specifies that the denominator is \textbf{Dollars}.

**Example 8.** When we want the number-phrase \textbf{+112 Dollars divided by} 4 copies of \(-2\), we say that

- the coefficient is \textbf{+112},
• the base from which we make the copies is $-2$,
• the plain exponent is 4

and we write the following staggered template in Dollars:

$\begin{align*}
+112 & \odot -2 \\
& \odot -2 \\
& \odot -2 \\
& \odot -2 \\
\end{align*}$

(1st division by $-2$)  
(2nd division by $-2$)  
(3rd division by $-2$)  
(4th division by $-2$)  

(Result of the repeated divisions)

The staggered template specifies what is to be done at each stage and therefore what the numerator of the result in Dollars will be.

3. Quite often, though, we will not want to get the actual result but just be able to discuss the repeated operations and, in that case, the use of staggered templates is cumbersome. So, what we will do is to let the boxes “go without saying” which will allow us to write an in-line template, that is:

i. For the numerators, we write on a single line:

   i. The coefficient,

   ii. The operation symbol followed by the 1st copy of the base

   iii. The operation symbol followed by the 2nd copy of the base

   iv. The operation symbol followed by the 3rd copy of the base

   v. Etc until all copies specified by the plain exponent have been written.

ii. For the denominator, we have a choice:

   - We can declare the denominator up front and then write the in-line template for the numerators,

   - We can write the in-line template for the numerators within square brackets and then write the denominator.

Example 9. Instead of writing the staggered template in Dollars


13.3 The Order of Operations

The use of in-line templates for repeated operations, though, poses a problem: how do we know for sure in what order the recipient of an in-line template is going to do the operations?
The reason this can be a problem is that this order can make all the
difference between the recipient arriving at the intended result and the re-
cipient arriving at something completely irrelevant.

1. When the operation being repeated is multiplication, it turns out that the
order in which the operations are done does not matter.

**Example 11.** Given the in-line template in Dollars

\[17 \times 2 \times 2 \times 2 \times 2 \times 2\]

the recipient might choose to compute it as

\[\begin{align*}
17 \times 2 \\
34 \times 2 \\
68 \times 2 \\
136 \times 2 \\
272 \times 2 \\
544 \times 2 \\
1088 \\
\end{align*}\]

or the recipient might choose to compute it as

\[\begin{align*}
2 \times 2 \\
2 \times 4 \\
2 \times 8 \\
2 \times 16 \\
2 \times 32 \\
17 \times 64 \\
1088 \\
\end{align*}\]

or as

\[\begin{align*}
2 \times 2 \\
4 \times 2 \\
8 \times 2 \\
2 \times 16 \\
2 \times 32 \\
17 \times 64 \\
1088 \\
\end{align*}\]

etc but, it does not matter as the result will always be 1088.

However, proving in general that the order in which the multiplications are
done does not matter takes some work because, as the number of copies gets
large, the number of ways in which the multiplications could be done gets
even larger and yet, to be able to make a general statement, we would have
to make sure that all of these ways have been accounted for. So, for the sake
of time, in the case of repeated *multiplications*, we will take the following for granted:

**THEOREM 6.** *The order in which* multiplications *are done does not matter.*

2. In the case of repeated *division*, though, the order usually makes a *huge* difference.

**EXAMPLE 12.** Given the in-line template in Dollars

\[
448 \div 2 \div 2 \div 2 \div 2 \div 2 \div 2
\]

and while the recipient might indeed choose to compute is as

\[
\begin{align*}
448 & \div 2 \\
224 & \div 2 \\
112 & \div 2 \\
56 & \div 2 \\
28 & \div 2 \\
14 & \div 2 \\
7 & 
\end{align*}
\]

the recipient might also choose to compute it as

\[
\begin{align*}
448 & \div 1 \\
2 & \div 2 \\
2 & \div 1 \\
2 & \div 2 \\
448 & \div 1
\end{align*}
\]

or as

\[
\begin{align*}
448 & \div 1 \\
2 & \div 2 \\
1 & \div 2 \\
0.5 & \div 2 \\
2 & \div 0.25 \\
448 & \div 0.25
\end{align*}
\]

etc

Thus, in the case of repeated *divisions* it is crucial to agree on the order in which to do them and so, in the absence of any instructions to that effect, we will use
DEFAULT RULE # 5. The order in which divisions are to be done is from left to right.

13.4 The Way to Powers

Eventually, we will devise a very powerful language to deal both with repeated multiplications and repeated divisions but, before we can do that, we need to clear the way.

1. While, as we have seen, 1 does tend to “go without saying”, what we can do when the coefficient in a repeated operation is 1 depends on whether the operation being repeated is multiplication or division.

a. When it is multiplication that is being repeated, we can let the coefficient 1 go without saying. However, the number of multiplications is then one less than the number of copies.

EXAMPLE 13. Given the in-line template in Dollars
\[ 1 \times 3 \times 3 \times 3 \times 3 \times 3 \]
we can write instead
\[ 3 \times 3 \times 3 \times 3 \times 3 \]
because we get 243 either way.
However, while we still have five copies of 3, we now have only four multiplications.

b. When it is division that is being repeated, we must write the coefficient 1 as, if we did not, we would be getting a different result.

EXAMPLE 14. Given the in-line template in Dollars
\[ 1 \div 2 \div 2 \div 2 \div 2 \div 2 \]
the 1 cannot go without saying because, while the given in-line template computes to \( \frac{1}{\text{2512}} \), if we don’t write the coefficient 1, we get an in-line template with coefficient 2 to be divided by four copies of 2:
\[ 2 \div 2 \div 2 \div 2 \div 2 \]
which computes to \( \frac{1}{\text{2}} \).

2. Repeated divisions are related to repeated multiplications. Indeed,
- instead of dividing a coefficient by a number of copies of the base,
- we can:
  i. multiply 1 repeatedly by the number of copies of the base,
  ii. divide the coefficient by the result of the repeated multiplication.

EXAMPLE 15. Given the in-line template in Dollars
\[ 448 \div 2 \div 2 \div 2 \div 2 \div 2 \]

\(^2\)Educologists will correctly point out that while \( 1 \times \) can “go without saying”, this is really where multiplication as a binary operation comes in.

\(^3\)Educologist will point out that, essentially, this is just the fact that, instead of dividing by a numerator, we can multiply by its reciprocal.
instead of computing it as follows:

\[
\begin{align*}
448 \div 2 &= 224 \\
224 \div 2 &= 112 \\
112 \div 2 &= 56 \\
56 \div 2 &= 28 \\
28 \div 2 &= 14 \\
14 \div 2 &= 7
\end{align*}
\]

we can proceed as follows:

i. We multiply 1 by the 6 copies of 2

\[
\begin{align*}
1 \times 2 &= 2 \\
2 \times 2 &= 4 \\
4 \times 2 &= 8 \\
8 \times 2 &= 16 \\
16 \times 2 &= 32 \\
32 \times 2 &= 64
\end{align*}
\]

ii. We divide the coefficient 448 by the result of this repeated multiplications:

\[
448 \div 64 = 7
\]

which indeed gives us the same result as the repeated division.

The advantage of this second way of computing in-line templates involving repeated divisions is that while we now have one more operation than we had divisions, the first multiplication, multiplying the coefficient 1 by the first copy of the base, is no work and, as we saw above, need in fact not even be written so that the number of operations requiring work is the same in both cases. But now all operations except one are multiplications which are a lot less work than divisions.

However, here again, proving in general that the results are always the same takes some work so that, for the sake of saving time, we will take for granted that:

**THEOREM 7.** A repeated division is the same as a single division of the coefficient by the result of 1 multiplied repeatedly by the same number of copies of the base.

\[
\text{Coefficient } \otimes \text{ copies } = \text{Coefficient } \otimes [1 \otimes \text{ copies}]
\]

3. In order to specify the second way of computing, we can write either:
A bracket in-line template where we write:
  i. The coefficient followed by a division symbol,
  ii. A pair of square brackets within which we write
  iii. 1 repeatedly multiplied by the same number of copies of the base.

**Example 16.** Instead of writing the in-line template in Dollars as

\[
+448 \div -2 \div -2 \div -2 \div -2 \div -2 \div -2
\]

we can write the bracket in-line template in Dollars as

\[
+448 \div \left[ +1 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \right]
\]

or as

\[
+448 \div \left[ -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \right]
\]

or

A fraction-like template where we write:
  i. The coefficient and, underneath,
  ii. A fraction bar and, underneath
  iii. 1 repeatedly multiplied by the same number of copies of the base
with the 1 able to “go without saying”.

underneath the repeated multiplication underneath the bar,

**Example 17.** Instead of writing the in-line template in Dollars

\[
+448 \div -2 \div -2 \div -2 \div -2 \div -2 \div -2
\]

we can write the in-line template in Dollars as

\[
\frac{+448}{+1 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2}
\]

or as

\[
\frac{+448}{-2 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2}
\]

**NOTE.** Whether we use a bracket in-line template or a fraction-like template, we need not write the 1 as, either way, there is something to remind us that the multiplications have to be done first:

- The square brackets
or
- The fraction bar

In general, though, we will prefer to use fraction-like templates with the 1 “going without saying”.

In other words, instead of:

\[
\text{Coefficient } \otimes \text{ copies } = \text{Coefficient } \otimes [1 \otimes \text{ copies}]
\]

we prefer to write
monomial
specifying-phrase
separator
signed exponent
superscript
signed power

\[
\text{Coefficient } \otimes \text{ copies } = \frac{\text{Coefficient}}{\text{copies}}
\]

but, even though both sides are read as

“Coefficient divided by copies”

- the division symbol \(\otimes\) on the left side of =

\[
\text{Coefficient } \otimes \text{ copies } = \frac{\text{Coefficient}}{\text{copies}}
\]
says that the coefficient is to be divided repeatedly by the copies of the base

- the fraction bar on the right side of =

\[
= \frac{\text{Coefficient}}{\text{copies}}
\]
says that the coefficient is to be divided by the result of the multiplication of 1 by the copies of the base.

13.5 Power Language

We are now ready to introduce a way of writing specifying-phrases that will work both for repeated multiplications and for repeated divisions.

1. The idea is to write just the coefficient, the base, the number of copies and whether the coefficient should be multiplied or divided by the copies. More precisely, in order to write a new kind of specifying-phrase which we will call a monomial specifying-phrase,

i. We write its numerator, that is we write:
   i. The coefficient,
   ii. The multiplication symbol \(\times\) or \(\otimes\) (depending on whether the numerators are plain or signed) as separator followed by the base,
       iii. A signed exponent, that is a signed numerator
           • whose sign is
             + when the coefficient is to be multiplied by the copies
             - when the coefficient is to be divided by the copies
           • whose size is the number of copies

In order to be separated from the base, the signed exponent must be written as a superscript, that is small and raised a bit above the base line.

ii. We write its denominator if it has not been declared up front.
The base together with the signed-exponent is called a signed power.

We then read monomial specifying-phrases as

“Coefficient multiplied/divided by number of copies of the base”
EXAMPLE 18. Given the in-line template in Dollars
\[ 17 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \]

- In order to write the monomial specifying-phrase,
  i. We write the coefficient 17:
  \[ 17 \]
  ii. We write the multiplication symbol \( \times \) as separator followed by the base 2:
  \[ 17 \times 2 \]
  iii. We write the signed exponent as a superscript with + to indicate that the coefficient is to be multiplied by the 6 copies of the base 2:
  \[ 17 \times 2^{+6} \]

- We read the monomial specifying-phrase
  \[ 17 \times 2^{+6} \]
  as
  \[ 17 \text{ multiplied by 6 copies of 2} \]

EXAMPLE 19. Given the in-line template
\[ 448 \div [2 \times 2 \times 2 \times 2 \times 2] \]

- In order to write the monomial specifying-phrase,
  i. We write the coefficient 448:
  \[ 448 \]
  ii. We write the multiplication symbol \( \times \) as separator followed by the base 2:
  \[ 448 \times 2 \]
  iii. We write the signed exponent with – to indicate that the coefficient is to be divided by 6 copies of the base 2:
  \[ 448 \times 2^{-6} \]

- We read the monomial specifying-phrase
  \[ 448 \times 2^{-6} \]
  as
  \[ 448 \text{ divided by 6 copies of 2} \]

NOTE. In other words, here, \( \times \) is really only a separator and has nothing to do with the kind of repeated operation we are specifying. While this way of writing things might seem rather strange, we will see in the next section how it turns out to make excellent sense.

2. As it happens, though, there is no procedure for identifying monomial specifying-phrases other than the procedures corresponding to staggered templates.

EXAMPLE 20. Given the following monomial specifying-phrase in Dollars
\[ 17 \times 2^{+6} \]

there is no way to identify it other than doing
\[ 17 \times 2^{+6} \]
\[ 34 \times 2 \]
This is in sharp contrast with the case of *repeated additions* for which there is a much shorter procedure for getting the result of repeated additions that is based on *multiplication* and with the case of *repeated subtractions* for which there is a much shorter procedure for getting the result based on *division*.

3. It is customary to distinguish monomial specifying-phrases in which the exponent has to be positive or 0 from monomial specifying-phrases in which the exponent can have any sign.

We will use the following names:

- A **Laurent monomial specifying-phrase** is a monomial specifying-phrases in which the exponent is a numerator that can have any sign.
- A **plain monomial specifying-phrase** is a monomial specifying-phrases in which the exponent is a numerator that can be only positive or 0 or, in other words, that can only be a plain numerator.