Chapter 15

Polynomials 1: Addition, Subtraction

While, as we saw in the preceding chapter, monomials behave very well with respect to *multiplication* and *division* in the sense that we can always multiply or divide a first monomial by a second monomial and get a monomial as a result, we will see that monomials behave very badly with respect to *addition* and *subtraction*. This, though, gives raise to a new type of *term* which will in fact play a fundamental role—to be described in the Epilogue at the end of this text—in the investigation of FUNCTIONS.

In the rest of this text, we will introduce and discuss the way this new type of terms behaves with respect to the four operations. These are the basics of what is called POLYNOMIAL ALGEBRA.

15.1 Monomials and Addition

We begin by looking at the way monomials behave with regard to *addition*. The short of it is that, most of the time, monomials *cannot* be added.

1. One way to look at why monomials usually cannot be added is to observe that powers are to monomials much the same as denominators are to number-phrases.

- Just like ordinary number-phrases need to involve the *same denominator* in order to be added, monomials need to involve the *same power* to be added.

  EXAMPLE 1. Just like

  $$17.52 \text{ Meters} + 4.84 \text{ Meters} = 22.36 \text{ Meters}$$

  we have that

  $$17.52x^{+6} + 4.84x^{+6} = 22.36x^{+6}$$

- Just like ordinary number-phrases that involve *different denominators* cannot be added and just make up a *combination*, monomials that involve *different powers* cannot be added and just make up a *combination.*

  EXAMPLE 2. Just like

  $$17.52 \textbf{ Feet} + 4.84 \textbf{ Inches} \text{ is a } \textit{combination}$$

  we have that

  $$17.52x^{+6} + 4.84x^{+4} \text{ is a } \textit{combination}$$

2. A more technical way to look at why monomials cannot be added when the powers are different is to try various ways of "adding" monomials and then to see what the results would be when we replace the *unspecified numerator* x by *specific* numerators.

EXAMPLE 3. Suppose we think that the rule for adding the monomials should be "add the coefficients and add the exponents".
Then, given for instance the monomials

$$+7x^{-2} \text{ and } -3x^{+3}$$

the rule "add the coefficients and add the exponents" would give us the following monomial as a result:

$$(+7 \oplus -3)x^{-2 \oplus +3}$$

that is

$$+4x^{+1}$$

Now while, on the one hand, there is no obvious reason why this should not be an acceptable result, on the other hand, monomials are waiting for x to be replaced by some specific numerator.
So, say we replace x by $+4$. The given monomials would then give:

$$\begin{aligned} +7x^{-2}\big|_{x:=+4} &= \frac{+7}{(+4) \bullet (+4)} \\ &= \frac{+7}{+16} \\ &= 0.4375 \end{aligned}$$

and

$$\begin{aligned} -3x^{+3}\big|_{x:=+4} &= -3 \bullet (+4) \bullet (+4) \bullet (+4) \\ &= -192 \end{aligned}$$

which, when we add them up, gives us

$$-191.5625$$

But, when we replace x by $+4$ in the supposed result, we get

$$+4x^{+1}\big|_{x:=+4} = +4 \bullet (+4)$$

$$= +16$$

Laurent polynomial reduced

So, in the end, the rule "add the coefficients and add the exponents" would not produce an acceptable result.

Even though, as it happens, no rule for adding monomials will survive replacement of x by a specific numerator, the reader is encouraged to try so as to convince her/him self that this is really the case.

15.2 Laurent Polynomials

A **Laurent polynomial** is a *combination* of *powers* involving:

- *exponents* that can be any *signed counting* numerator (including 0).
- *coefficients* that can be any *signed decimal* numerator

EXAMPLE 4. All of the following are Laurent polynomials:

$$+22.71x^{+3} + 0.3x^{0} - 47.03x^{+2} + 57.89x^{-3}$$
$$+21.09x^{-4} - 33.99x^{+2} + 45.02x^{-1} + 52.74x^{+1} - 34.82x^{+7}$$
$$-30.18x^{+6} - 41.02x^{+5} + 5.6x^{+4}$$
$$+20.13x^{+3} + 0.03x^{+5} + 50.01x^{0} - 0.04x^{+1}$$
$$-0.02x^{-7} + 18.03x^{+6}$$

1. While there is nothing difficult about what Laurent polynomials *are*, we need to agree on a few rules to make them easier to *work* with since, otherwise, it is not always easy even just to see if two Laurent polynomials are the same or not.

EXAMPLE 5. The following two Laurent polynomials are the same

$$+0.3x^{0} - 47.03x^{+2} + 22.71x^{+3} + 57.89x^{-3}$$
$$+57.89x^{-3} + 22.71x^{+3} + 0.3x^{0} - 47.03x^{+2}$$

but the following two Laurent polynomials are not the same

$$+0.3x^{0} - 47.03x^{+2} - 22.71x^{+3} + 57.89x^{-3}$$
$$+57.89x^{-3} + 22.71x^{+3} + 0.3x^{0} - 47.03x^{+2}$$

EXAMPLE 6. The following two Laurent polynomials are in fact the same

$$+2x^{+3} + 6x^{-4}$$
$$-6x^{+3} + 4x^{-4} + 8x^{+3} + 2x^{-4}$$

a. The first thing we have to agree on is that Laurent polynomials must always be **reduced**, that is that monomials in a given Laurent polynomial that *can* be added (because they involve the same power) *must* in fact be added.

EXAMPLE 7. Given the following Laurent polynomial

$$-6x^{+3} + 4x^{-4} + 8x^{+3} + 2x^{-4}$$

it must be *reduced* to

ascending order of exponents
descending order of exponents

$$+2x^{+3} + 6x^{-4}$$

before we do anything else.

b. The second thing we have to do is to agree on some order in which to write the monomials in a Laurent polynomial.

i. We will agree that:
The monomials in a Laurent polynomial will and can only be written in either one of two orders:

- **ascending order of exponents**, that is, as we read or write a Laurent polynomial from left to right, the *exponents* must get *larger and larger* regardless of the *coefficients*.
- **descending order of exponents**, that is, as we read or write a Laurent polynomial from left to right, the *exponents* must get *smaller and smaller* regardless of the *coefficients*.

EXAMPLE 8. The following Laurent polynomial

$$-47.03x^{+2} + 57.89x^{-3} + 22.71x^{+4} + 0.3x^{0}$$

can only be written either in *ascending* order of *exponents*

$$+57.89^{-3} + 0.3^{0} - 47.03x^{+2} + 22.71x^{+4}$$

or in *descending* order of *exponents*

$$+22.71x^{+4} - 47.03x^{+2} + 0.3x^{0} + 57.89x^{-3}$$

regardless of the *coefficients*.

ii. Which of the two orders is to be used depends on the *size* of the numerators with which x can be replaced:

- The *ascending* order must be used when x can be replaced only by *small* numerators,
- The *descending* order must be used when when x can be replaced only by for *large* numerators.

We will see the reason in a short while.

NOTE. When the size of what x stands for is *unknown*, it is customary, even if for no special reason, to use the *descending* order of exponents.

c. The third thing we have to do is to introduce *customary practices* even though these practices will not be followed here.

i. It is usual to write just *plain* exponents instead of *positive* exponents.

EXAMPLE 9. Instead of writing

$$+57.89x^{-3} + 0.3x^{0} - 47.03x^{+2} + 22.71x^{+4}$$

it is usual to write

$$+57.89x^{-3} + 0.3x^{0} - 47.03x^{2} + 22.71x^{4}$$

ii. It is usual not to write the exponent $+1$ at all.

EXAMPLE 10. Instead of writing

$$+57.89x^{+3} + 0.3x^{+2} - 47.03x^{+1} + 29.77x^{+4}$$

consecutive
missing power
evaluate

it is usual to write

$$+57.89x^{+3} + 0.3x^{+2} - 47.03x \quad + 29.77x^{+4}$$

iii. It is usual not to write the power x^0 at all.

EXAMPLE 11. Instead of

$$+57.89x^{-3} + 0.3x^{0} - 47.03x^{+2} + 22.71x^{+4}$$

it is usual to write

$$+57.89x^{-3} + 0.3 \quad - 47.03x^{+2} + 22.71x^{+4}$$

iv. Most of the time, the exponents of the powers will be **consecutive** but occasionally there can be **missing powers**.

EXAMPLE 12. The following Laurent polynomials in which the powers are *consecutive* are fairly typical of those that we will usually encounter.

$$-47.03x^{+3} + 57.89x^{+2} + 22.71x^{+1} + 0.3x^{0}$$
$$-47.03x^{+1} + 57.89x^{0} + 22.71x^{-1}$$
$$-47.03x^{-1} + 57.89x^{0} + 22.71x^{+1} + 0.3x^{+2}$$

EXAMPLE 13. The following Laurent polynomials in which at least one power is *missing* are fairly typical of those that we will occasionally encounter.

$$-47.03x^{+3} + 0.3x^{0}$$
$$-47.03x^{+2} + 57.89x^{0} + 22.71x^{-1}$$
$$-47.03x^{-1} + 57.89x^{0} + 22.71x^{+1} + 0.3x^{+3}$$

When working with a Laurent polynomial in which powers are *missing*, it is much safer to insert in their place powers with coefficient 0.

EXAMPLE 14. Instead of working with

$$-47.03x^{+3} + 13.3x^{0}$$

it is much safer to work with

$$-47.03x^{+3} + 0x^{+2} + 0x^{+1} + 13.3x^{0}$$

2. Laurent polynomials are specifying-phrases and we **evaluate** Laurent polynomials in the usual manner, that is we replace x by the required numerator and we then compute the result.

a. EXAMPLE 15. Given the Laurent polynomial

$$-47.03x^{+2} \oplus +13.3x^{-3}$$

when $x := -5$

$$\begin{aligned} -47.03x^{+2} \oplus +13.3x^{-3}\big|_{x:=-5} &= -47.03(-5)^{+2} \oplus +13.3(-5)^{-3} \\ &= [-47.03 \otimes (-5)(-5)] \oplus \left[\frac{+13.3}{(-5)(-5)(-5)}\right] \\ &= [-47.03 \otimes +25] \oplus \left[\frac{+13.3}{-125}\right] \\ &= -1175.75 \oplus +0.1064 \\ &= -1175.6436 \end{aligned}$$

diminishing
plain polynomial

b. When the coefficients are all single-digit counting numerators and we replace x by TEN, the result shows an interesting connection between Laurent polynomials and decimal numbers.

EXAMPLE 16. Given the Laurent polynomial

$$4x^{+3} + 7x^{+2} + 9x^{+1} + 8x^{0} + 2x^{-1} + 5x^{-2} + 6x^{-3}$$

when $x := 10$ we get:

$$\begin{aligned}
4x^{+3} + 7x^{+2} + 9x^{+1} + 4x^{0} + 2x^{-1} + 7x^{-2} + 7x^{-3}\Big|_{x=10} &= \\
&= 4 \times 10^{+3} + 7 \times 10^{+2} + 9 \times 10x^{+1} + 8 \times 10^{0} + 2 \times 10^{-1} + 5 \times 10^{-2} + 6 \times 10^{-3} \\
&= 4 \times 1000. + 7 \times 100. + 9 \times 10. + 8 \times 1. + 2 \times 0.1 + 5 \times 0.01 + 6 \times 0.001 \\
&= 4000. + 700. + 90. + 8. + 0.2 + 0.05 + 0.006 \\
&= 4\ 7\ 9\ 8.\ 2\ 5\ 6
\end{aligned}$$

which is the decimal number whose digits are the coefficients of the Laurent polynomial.

3. We are now in a position at least to state the reason for allowing only the *ascending* order of exponents and the *descending* order of exponents:

When we replace x by a specific numerator and go about evaluating the Laurent polynomial, we evaluate, one by one, each one of the monomials in the Laurent polynomial. But what happens is that

- When x is replaced by a numerator that is *large in size*, the more copies there are in a monomial, the *larger in size* the result will be.
- When x is replaced by a numerator that is *small in size*, the more copies there are in a monomial, the *smaller in size* the result will be.

But what we want, no matter what, is that the size of the successive results go **diminishing**. So,

- When x is to be replaced by a numerator that is going to be *large in size*, we will want the Laurent polynomial to be written in *descending order of exponents*.
- When x is to be replaced by a numerator that is going to be *small in size*, we will want the Laurent polynomial to be written in *ascending order of exponents*.

For lack of time, we cannot go here into any more detail but the interested reader will find this discussed at some length in the Epilogue.

15.3 Plain Polynomials

A **plain polynomial** is a combination of *powers* involving:

- *exponents* that can be any *positive* counting numerator as well as 0.
- *coefficients* that can be any signed decimal numerator

In other words, a *plain* polynomial is a combination of *powers* that do not involve any *negative* exponent—but can involve the exponent 0. polynomial

Example 17. The following are *plain* polynomials:

$$-47.03x^{+3} + 57.89x^{+2} + 22.71x^{+1} + 0.3x^{0}$$
$$0.3x^{0} - 47.03x^{+1} + 57.89x^{+2} + 22.71x^{+3}$$

The following are *not plain* polynomials:

$$-47.03x^{+3} + 57.89x^{+2} + 22.71x^{+1} + 0.3x^{0} - 22.43x^{-1}$$
$$-22.43x^{-1} + 0.3x^{0} - 47.03x^{+1} + 57.89x^{+2} + 22.71x^{+3}$$

1. When we replace x by TEN in a *plain* polynomial whose coefficients are all single-digit counting numerators, the result is a *counting* number.

Example 18. Given the *plain* polynomial

$$4x^{+3} + 7x^{+2} + 9x^{+1} + 8x^{0}$$

when $x := 10$ we get:

$$\begin{aligned} 4x^{+3} + 7x^{+2} + 9x^{+1} + 4x^{0}\Big|_{x:=10} &= 4 \times 10^{+3} + 7 \times 10^{+2} + 9 \times 10x^{+1} + 8 \times 10^{0} \\ &= 4 \times 1000 + 7 \times 100 + 9 \times 10 + 8 \times 1 \\ &= 4000 + 700 + 90 + 8 \\ &= 4\ 7\ 9\ 8 \end{aligned}$$

which is the *counting* number whose digits are the coefficients of the *plain* polynomial.

2. Just like decimal numerators are not really more difficult to use than just counting numerators—they just require understanding that the decimal point indicates which of the digits in the decimal numerator corresponds to the denominator[1], Laurent polynomials are just as easy to use as just plain polynomials. This is particularly the case since, in the case of polynomials, we do not have to worry about the "place" of a monomial in a polynomial since the place is always given by the exponent

3. Just like decimal numbers are vastly more useful than just counting numbers, Laurent polynomials will be vastly more useful than plain polynomials for our purposes as the discussion in the EPILOGUE will show.

4. Since, from the point of view of handling them, there is not going to be any difference between Laurent polynomials and plain polynomials, we will just the word **polynomial**.

[1]But then of course, since Educologists have a deep aversion to denominators, they are sure to disagree.

add
like monomials
⊞
addition of polynomials

15.4 Addition

Just like *combinations* can always be added to give another combination, *polynomials* can always be added to give another polynomial.

EXAMPLE 19. Just like the combinations

17 **Apples** & 4 **Bananas** and 7 **Bananas** & 8 **Carrots**

can be added to give another combination:

17 **Apples** &	4 **Bananas**	
	7 **Bananas** &	8 **Carrots**
17 **Apples** &	11 **Bananas** &	8 **Carrots**

the polynomials

$$-17x^{+6} + 4x^{-3} \quad \text{and} \quad +7x^{-3} + 8x^{+2}$$

can be added to give another polynomial:

$$\begin{array}{lll} -17x^{+6} & +4x^{-3} & \\ & +7x^{-3} & +8x^{+2} \\ \hline -17x^{+6} & +11x^{-3} & +8x^{+2} \end{array}$$

1. To **add** two polynomials with signed coefficients, we *oplus* the coefficients of **like monomials** that is of monomials with the same exponent. We will use the symbol ⊞ to write the specifying-phrase that corresponds to the **addition of polynomials**.

EXAMPLE 20. Given the polynomials

$$-17x^{+6} + 4x^{-3} \quad \text{and} \quad +7x^{-3} + 8x^{+2}$$

the specifying-phrase for addition will be

$$-17x^{+6} + 4x^{-3} \boxplus +7x^{-3} + 8x^{+2}$$

and to identify it, we will write

$$\begin{aligned} -17x^{+6} + 4x^{-3} \boxplus +7x^{-3} + 8x^{+2} &= -17x^{+6} + [+4 \oplus +7]x^{-3} + 8x^{+2} \\ &= -17x^{+6} + 11x^{-3} + 8x^{+2} \end{aligned}$$

2. The only difficulties when adding polynomials occur when one is not careful to write them:

- in order—whether ascending or descending
- with missing monomials written-in with 0 coefficient

EXAMPLE 21. Given the polynomials

$$-17x^{+3} - 14x^{+2} - 8x^{0} + 4x^{-1} \quad \text{and} \quad +7x^{+4} + 8x^{+3} - 11x^{+1} - 4x^{-2}$$

consider the difference between the following two ways to write the addition of two polynomials:

- When we *do not* write the polynomials in order and *do not* write-in missing monomials with a 0 coefficient, we get:

$$\begin{array}{l} -17x^{+3} - 14x^{+2} \quad - 8x^{0} \quad + 4x^{-1} \\ \ +7x^{+4} \quad + 8x^{+3} - 11x^{+1} - 4x^{-2} \\ \hline \end{array}$$

  and it is not easy to do the addition and get the result:

$$+7x^{+4} - 9x^{+3} - 14x^{+2} - 11x^{+1} - 8x^{0} + 4x^{-1} - 4x^{-2}$$

- When we *do* write the polynomials in order and we *do* write-in the missing monomials with a 0 coefficient, we get:

$$\begin{array}{rrrrrrr} 0x^{+4} & - 17x^{+3} & - 14x^{+2} & + 0x^{+1} & - 8x^{0} & + 4x^{-1} & + 0x^{-2} \\ +7x^{+4} & + 8x^{+3} & + 0x^{+2} & - 11x^{+1} & + 0x^{0} & + 0x^{-1} & - 4x^{-2} \\ \hline +7x^{+4} & - 9x^{+3} & - 14x^{+2} & - 11x^{+1} & - 8x^{0} & + 4x^{-1} & - 4x^{-2} \end{array}$$

  where the result is much easier to get.

3. One way in which *polynomials* are easier than *numerators* to deal with is that when we add them there is no so-called "carry-over".

The reason we have "carry-over" in ARITHMETIC is that when dealing with combinations of powers of TEN, the coefficients can only be *digits*. So, when we add, say, the hundreds, if the result is still a single digit, we can can write it down but if the result is more than *nine*, we have no single digit to write the result down and we must *change* TEN of the hundreds for a thousand which is what the "carry-over" is.

But in ALGEBRA, with combinations of powers of x, there is no such restriction on the coefficients which can be any numerator and so, when we add, we can write down the result whatever it is.

EXAMPLE 22.

- When we add the numerators 756.92 and 485.57 we get:

$$\begin{array}{r} {\scriptstyle 1\,1\,1\ \ \ \ \ \ \ } \\ 756.92 \\ +\ \ 485.57 \\ \hline 1242.49 \end{array}$$

  in which there are three "carry-overs" because there are three places where we couldn't write the result with a single digit.

- When we add the corresponding single-digit coefficient polynomials, we get:

$$\begin{array}{rrrrrr} & +7x^{+2} & + 5x^{+1} & + 6x^{0} & + 9x^{-1} & + 2x^{-2} \\ \boxplus & +4x^{+2} & + 8x^{+1} & + 5x^{0} & + 5x^{-1} & + 7x^{-2} \\ \hline \end{array}$$

$$+11x^{+2} + 13x^{+1} + 11x^{0} + 14x^{-1} + 9x^{-2}$$

in which there is no" carry-over" since we can write two-digit coefficients.

15.5 Subtraction

Subtraction "works" essentially the same way as addition except of course that while, in the case of *addition*, we *oplus* the monomials of the second polynomial, in the case of *subtraction*, we *ominus* the monomials of the second polynomial, that is we *oplus the opposite* of the monomials of the second polynomial.

EXAMPLE 23. In order to subtract the second polynomial from the first:

$$\begin{array}{cccccc} & +2x^{+2} & +4x^{+1} & +6x^{0} & -6x^{-1} & -5x^{-2} \\ \boxminus & -9\,x^{+2} & -3\,x^{+1} & +3\,x^{0} & -5\,x^{-1} & +7\,x^{-2} \\ \hline \end{array}$$

we add the opposite of the second polynomial to the first polynomial, that is we *oplus the opposite* of each monomial in the second polynomial to the corresponding monomial in the first polynomial:

$$\begin{array}{cccccc} & +2x^{+2} & +4x^{+1} & +6x^{0} & -6x^{-1} & -5x^{-2} \\ \boxplus & +9\,x^{+2} & +3\,x^{+1} & -3\,x^{0} & +5\,x^{-1} & -7\,x^{-2} \\ \hline & +11x^{+2} & +7x^{+1} & +3x^{0} & -1x^{-1} & -12x^{-2} \end{array}$$

Here again, things are easier with polynomials than with numerals since there is no "borrowing".