near
small
binomial

Chapter 17

Polynomials 3: Powers of $x_0 + h$

While it is easy to compute with powers of a *counting*-numerator, it is a lot more difficult to compute with powers of a *decimal*-numerator.

EXAMPLE 1. While it is easy to find that:

$$5 \bullet 3^4 = 405$$

it is a lot more difficult to find that

$$4 \bullet 3.14^4 = 388.84684864$$

But the main issue is that the result of a *repeated-multiplication* with a *base* that is a decimal numerator will usually involve a lot more decimals than are in the base and than we really want so that a lot of the work is wasted.

EXAMPLE 2. In

$$4 \bullet 3.14^4 = 388.84684864$$

the base, 3.14, has only two decimals but the result, 388.84684864, most probably has a lot more decimals than we want.

In this chapter we will investigate a procedure that will allow us to get only the number of decimals we want. It is based on the fact that any *decimal* numerator is always **near** a *counting* numerator in the sense that any *decimal* numerator is equal to a *counting*-numerator plus a **small** numerator

EXAMPLE 3. 3.14 is *near* 3 because $3.14 = 3 + 0.14$ and 0.14 is *small*

We will thus investigate the powers of the **binomial** x_0+h. We will begin by investigating the case in which the repeated multiplication involves two copies of the *binomial* and then the case in which the repeated multiplication involves three copies of the *binomial*. Then we will develop a procedure for the cases in which the repeated multiplication involves at least three copies of the *binomial*.

17.1 The Second Power: $(x_0 + h)^2$

1. In this case, the *repeated-multiplication procedure* is simple enough. In order to compute the second power of $x_0 + h$, we write, keeping in mind that we want the monomials to appear in order of diminishing sizes and since x_0 and h both stand for *signed* numerators:

$$\begin{array}{ccccc} & & x_0 & \oplus & h \\ & & x_0 & \oplus & h \\ \hline & & x_0 h & \oplus & h^2 \\ x_0^2 & \oplus & x_0 h & & \\ \hline x_0^2 & \oplus & 2x_0 h & \oplus & h^2 \end{array}$$

a. We begin by looking at what happens in ARITHMETIC which is that the multiplication procedure essentially keeps track and respects the sizes—but, because of carryovers, only roughly so.

EXAMPLE 4. In order to compute 3.2^2, we actually compute $(3 + 0.2)^2$ and write—since we are dealing with *plain* numerators:

$$\begin{array}{ccccc} & & 3 & + & 0.2 \\ & & 3 & + & 0.2 \\ \hline & & 3 \bullet 0.2 & + & 0.2^2 \\ 3^2 & + & 3 \bullet 0.2 & & \\ \hline 3^2 & + & 2 \bullet 3 \bullet 0.2 & + & 0.2^2 \end{array}$$

that is

$$\begin{array}{ccccc} & & 3 & + & 0.2 \\ & & 3 & + & 0.2 \\ \hline & & 0.6 & + & 0.04 \\ 9 & + & 0.6 & & \\ \hline 9 & + & 1.2 & + & 0.04 \end{array}$$

The multiplication procedure kept roughly track of the sizes except for what the carry-over caused:

- All the way to the left are the "ones"
- In the middle are the "tenths"
- All the way to the right are the "hundredths"

so that if we want:

- No decimal, we write
$$3.2^2 = 10 + (...)$$
- One decimal, we write
$$3.2^2 = 10.2 + (...)$$

- Two decimals, we write

$$3.2^2 = 10.24$$

where $+\ (...)$ is there to remind us that we are ignoring something too "in the tenths" to matter here.

EXAMPLE 5. In order to compute 2.8^2, we observe that 2.8 is nearer 3 than 2 so that we actually compute $(3 \oplus -0.2)^2$ and write—since we are now dealing with *signed* numerators:

$$\begin{array}{ccccc} & & +3 & \oplus & -0.2 \\ & & +3 & \oplus & -0.2 \\ \hline & & +3 \bullet -0.2 & \oplus & (-0.2)^2 \\ (+3)^2 & \oplus & +3 \bullet -0.2 & & \\ \hline (+3)^2 & \oplus & 2 \bullet +3 \bullet -0.2 & \oplus & (-0.2)^2 \end{array}$$

that is

$$\begin{array}{ccccc} & & +3 & \oplus & -0.2 \\ & & +3 & \oplus & -0.2 \\ \hline & & -0.6 & \oplus & +0.04 \\ +9 & \oplus & -0.6 & & \\ \hline +9 & \oplus & -1.2 & \oplus & +0.04 \end{array}$$

The multiplication procedure kept roughly track of the sizes except for what the carry-over caused:

- All the way to the left are the "ones"
- In the middle are the "tenths"
- All the way to the right are the "hundredths"

so that if we want:

- No decimal, we write

$$2.8^2 = 8 + (...)$$

- One decimal, we write

$$2.8^2 = 7.8 + (...)$$

- Two decimals, we write

$$2.8^2 = 7.84$$

where $+\ (...)$ is there to remind us that we are ignoring something, *positive or negative*, too "in the tenths" to matter here.

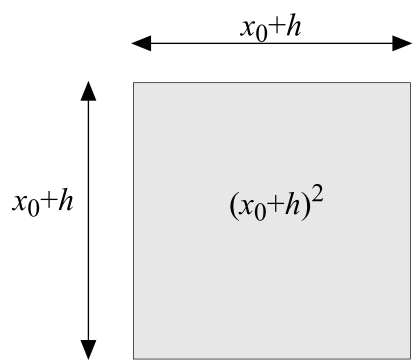
b. In algebra, a very frequent case is when we want a template for the power of any decimal-numerator in the neighborhood of a given x_0. In other words, we do not want yet to commit ourselves to how far the decimal-numerator is from the given x_0 and we use h to represent how far the decimal-numerator is from the given x_0.

Of course, when, ultimately, we replace h by some "in the tenths" number, there remains the possibility that a carryover will mess up the result a little bit. This though is something that we will not deal with here. (But see the Epilogue.)

EXAMPLE 6. In order to get a template for the second power of any decimal-numerator near 3, both above 3 and below 3, we write:

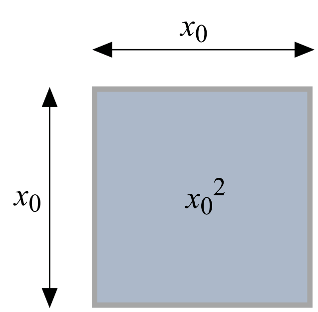
		3	$\oplus$	h
		3	$\oplus$	h
		$3h$	$\oplus$	h^2
3^2	$\oplus$	$3h$		
3^2	$\oplus$	$2 \bullet 3h$	$\oplus$	h^2

2. Another, much more fruitful to get the above template is to go back to the definition of multiplication in terms of the *area of a rectangle* so that $(x_0 + h)^2$ is the area of a $x_0 + h$ by $x_0 + h$ *square*:

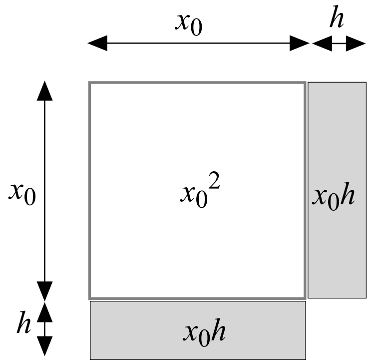


What we then do is to enlarge the sides of a x_0 by x_0 square by h but, for the sake of clarity, we will construct the enlarged square one step at a time:

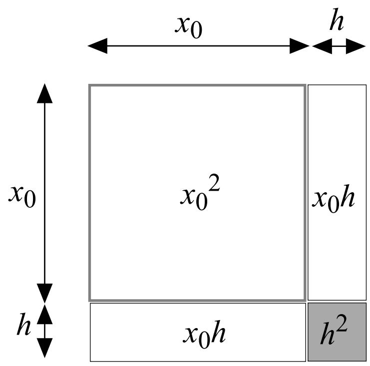
i. We start with x_0^2 as the area of a square with side x_0:



ii. We now enlarge the sides of the square by h in each dimension which adds two $x_0 + h$ by h rectangles:



iii. We complete the enlarged square by adding one h by h square:



EXAMPLE 7. In order to get a template to get the second power of any decimal-numerator near 3, both above 3 and below 3, we visualize the above picture and see in our mind that we need the area of:
i. the original square: 3^2
ii. the two rectangular strips: $2 \bullet 3 \bullet h$
iii. the little square: h^2
so that we have the template:

$$(3 \oplus h)^2 = 3^2 \oplus 2 \bullet 3 \bullet h \oplus h^2$$

This second approach has three major advantages over the first one:
i. The terms in the sum are clearly in order of *diminishing size*: Since x_0 is "in the ones" and h is "in the tenths",

- both dimensions of the "initial square" are "in the ones" so that x_0^2 is "in the ones",
- one dimension of the rectangles is "in the ones" but the other dimension is "in the tenths" so that $2x_0h$ is "in the tenths",
- both dimensions of the "little square" are "in the tenths" so that h^2 is "in the hundredths".

ii. When we will need formulas for $(x_0 + h)^3$, $(x_0 + h)^4$, etc, not only will repeated multiplication get out of hand but, as we shall see, we will never

need more than the first few monomials in the result.
iii. If all we need is a particular monomial in the result, which is often the case, we can get it from the picture without having to do the whole repeated multiplication.

EXAMPLE 8. If, for whatever reason, we need the h monomial in $(3 \oplus h)^2$, we visualize the two rectangular strips and we write:

$$2 \bullet 3 \bullet h$$

THEOREM 9 (Addition Formula for Quadratics).

$$(x_0 + h)^2 = x_0^2 + 2x_0h + h^2$$

17.2 The Third Power: $(x_0 + h)^3$

For the sake of brevity we omit the investigation of what happens in arithmetic.

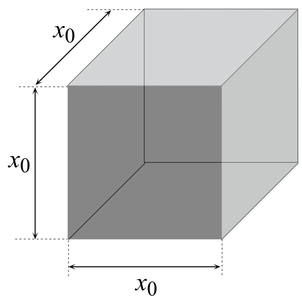
1. The repeated-multiplication procedure already begins to be painful: First we must multiply two copies of $x_0 + h$:

$$\begin{array}{ccccc} & & x_0 & + & h \\ & & x_0 & + & h \\ \hline & & x_0h & + & h^2 \\ x_0^2 & + & x_0h & & \\ \hline x_0^2 & + & 2x_0h & + & h^2 \end{array}$$

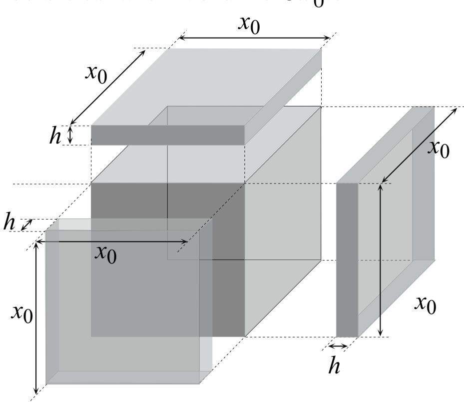
Then, we must multiply $x_0^2 + 2x_0h + h^2$ by the third copy of $x_0 + h$

$$\begin{array}{ccccccc} & & x_0^2 & + & 2x_0h & + & h^2 \\ & & & & x_0 & + & h \\ \hline & & x_0^2h & + & 2x_0h^2 & + & h^3 \\ x_0^3 & + & 2x_0^2h & + & x_0h^2 & & \\ \hline x_0^3 & + & 3x_0^2h & + & 3x_0h^2 & + & h^3 \end{array}$$

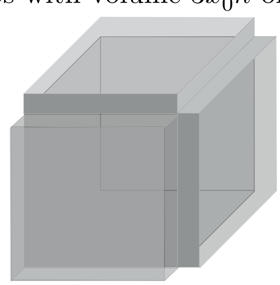
2. Another, much more fruitful approach to the addition formula is to go back to the definition of multiplication in terms of the area/volume of a rectangle so that $(x_0 + h)^3$ is the volume of a $x_0 + h$ by $x_0 + h$ by $x_0 + h$ cube:
What we then do is to enlarge the three sides of a x_0 by x_0 cube by h but, for the sake of clarity, we will construct the enlarged cube one step at a time:
i. We draw the initial cube with volume x_0^3:



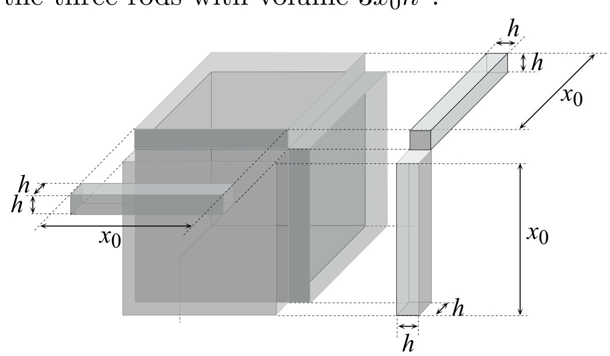
ii. We draw the three slabs with volume $3x_0^2h$:



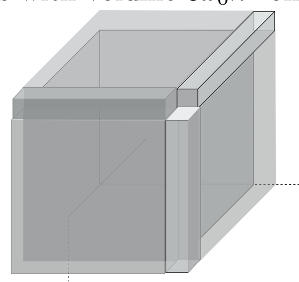
iii. We glue the three slabs with volume $3x_0^2h$ onto what we already glued:



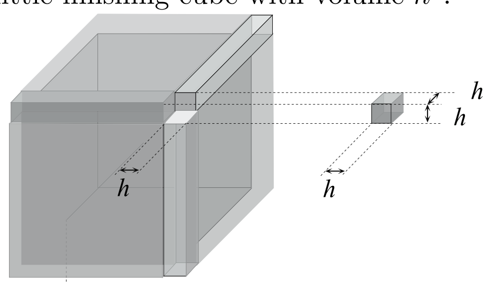
iv. We draw the three rods with volume $3x_0h^2$:



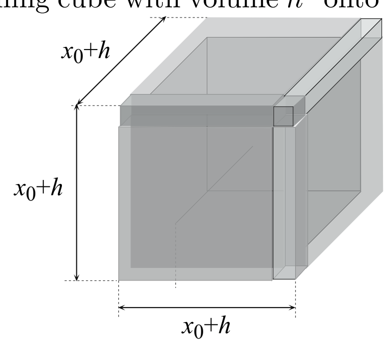
v. We glue the three rods with volume $3x_0h^2$ onto what we already glued:



vi. We draw the little finishing cube with volume h^3:



We glue the little finishing cube with volume h^3 onto what we already glued:



This approach has three major advantages over the *repeated-multiplication*:

i. The terms in the sum are in *order of diminishing size.* Since x_0 is "in the ones" and h is "in the tenths",

- all three dimensions of the "initial cube" are "in the ones" so that x_0^3 is "in the ones",
- two dimensions of the "slabs" are "in the ones" but the third dimension is "in the tenths" so that, if h is "in the tenths", then $3x_0^2h$ is "in the tenths",
- one dimensions of the "square rods" is "in the ones" so that, if h is "in the tenths", then $3x_0h^2$ is "in the hundredths".
- all three dimensions of the "little cube" are "in the tenths" so that, if h is "in the tenths", then x_0h^3 is "in the thousandths".

ii. If all we need is a particular one of the terms, which will often be the case, we can get it from the picture without having to do the whole multiplication. pattern
iii. Later on, when we shall need formulas for $(x_0 + h)^4$, etc, not only will repeated multiplication get out of hand but, as we shall see, we will never need more than the first few monomials of the result.
THEOREM 10 (ADDITION FORMULA for CUBICS).

$$(x_0 + h)^3 = x_0^3 + 3x_0^2 h + 3x_0 h^2 + h^3$$

17.3 Higher Powers: $(x_0 + h)^n$ when $n > 3$

Here of course:
- Repeated-multiplication is of course going to be ever more painful
- We cannot make pictures because we would need to be able to draw in more than 3 dimensions.

So, we need to find a *procedure*.

1. We begin by looking for a **pattern** in what we have so far. In order to see better what we are doing, we will not let anything go without saying.
a. When the exponent is 3, we had:

$$\begin{aligned}(x_0 + h)^3 &= x_0^3 + 3x_0^2 h + 3x_0 h^2 + h^3 \\ &= x_0^3 h^0 + 3x_0^2 h^1 + 3x_0^1 h^2 + x_0^0 h^3 \\ &= x_0 \bullet x_0 \bullet x_0 \ \& \ 3 \bullet x_0 \bullet x_0 \bullet h \ \& \ 3 \bullet x_0 h \bullet h \ \& \ h \bullet h \bullet h \\ &= 1 \bullet x_0 \bullet x_0 \bullet x_0 \ \& \ 3 \bullet x_0 \bullet x_0 \bullet h \ \& \ 3 \bullet x_0 h \bullet h \ \& \ 1 \bullet h \bullet h \bullet h\end{aligned}$$

Looking at the *factors* and the *coefficients* separately, we get the following:
- The *factors* are

  $$x_0 \bullet x_0 \bullet x_0 \qquad x_0 \bullet x_0 \bullet h \qquad x_0 \bullet h \bullet h \qquad h \bullet h \bullet h$$

  In other words, starting with 3 copies of x_0

  $$x_0 \bullet x_0 \bullet x_0$$

  we get the others by replacing one by one the copies of x_0 by copies of h.
- The *coefficients* are

  $$1 \qquad 3 \qquad 3 \qquad 1$$

  Here we cannot see the pattern,

b. When the exponent is 2, we have

$$\begin{aligned}(x_0 + h)^2 &= x_0^2 + 2x_0 h + h^2 \\ &= x_0^2 h^0 + 2x_0^1 h^1 + x_0^0 h^2 \\ &= x_0 \bullet x_0 \ \& \ 2 \bullet x_0 \bullet h \ \& \ h \bullet h\end{aligned}$$

$$= 1 \bullet x_0 \bullet x_0 \ \& \ 2 \bullet x_0 \bullet h \ \& \ 1 \bullet h \bullet h$$

Looking at the *factors* and the *coefficients* separately, we get the following:

- The *factors* are
$$x_0 \bullet x_0 \qquad x_0 \bullet h \qquad h \bullet h$$
In other words, starting with
$$x_0 \bullet x_0$$
we get the others by replacing one by one the copies of x_0 by copies of h.
- The *coefficients* are
$$1 \qquad 2 \qquad 1$$
Here again we cannot see the pattern.

c. When the exponent is 1, we have

$$\begin{aligned}(x_0 + h)^1 &= x_0 + h \\ &= x_0^1 + h^1 \\ &= x_0^1 h^0 + x_0^0 h^1 \\ &= x_0 \ \& \ h \\ &= 1 \bullet x_0 \ \& \ 1 \bullet h\end{aligned}$$

Looking at the *factors* and the *coefficients* separately, we get the following:

- The *factors* are
$$x_0 \qquad h$$
In other words, starting with
$$x_0$$
we get the others by replacing the one copy of x_0 by a copy of h.
- The *coefficients* are
$$1 \qquad 1$$
Here we cannot see the pattern,

2. Putting everything together, though,

- The procedure for finding the *powers* seems to be in all cases:
  i. Make as many copies of x_0 as what the exponent n in $(x_0 + h)^n$ indicates
  ii. Make as many copies plus 1 of what the exponent n in $(x_0 + h)^n$ indicates
  iii. Starting with the second copy, replace one by one the copies of x_0 by copies of h
- In order to see a pattern for the *coefficients*, we write them starting with exponent 1 and ending with exponent 3:

$$\begin{array}{ccccccc} & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

The way things are arranged, we see that we get each entry in what is called the **PASCAL TRIANGLE** by adding its two **parent-entries** that is the two entries just above it.

PASCAL TRIANGLE
parent-entries

Thus, the next line in the PASCAL TRIANGLE would be:

1 4 6 4 1

3. *Proving* that all this is indeed the case would involve more work than we are willing to do here and so we will take the following for granted:

THEOREM 11 (BINOMIAL THEOREM). *The addition formula for a binomial of degree n is:*

$$\begin{aligned}(x_0 + h)^n = x_0^n h^0 + \frac{n}{1} x_0^{n-1} h^{+1} &+ \frac{n(n-1)}{1 \cdot 2} x_0^{n-2} h^{+2} \\ &+ \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x_0^{n-3} h^{+3} \\ &+ \dots \\ &+ \frac{n(n-1)(n-2)\cdots(1)}{1 \cdot 2 \cdot 3 \cdots n} x_0^0 h^n\end{aligned}$$

4. According to the **BINOMIAL THEOREM**,

$$\begin{aligned}(x_0 + h)^0 &= x_0^0 h^0 \\ &= 1\end{aligned}$$

which is of course as it should be. Moreover, since the coefficient 1 goes without saying, this means that the very first line in the PASCAL TRIANGLE is 1 so that the "complete" PASCAL TRIANGLE is:

$n := 0$								1						
$n := 1$							1		**1**					
$n := 2$						1		**2**		1				
$n := 3$					1		**3**		3		1			
$n := 4$				1		**4**		6		4		1		
$n := 5$			1		**5**		10		10		5		1	
$n := 6$		1		**6**		15		20		15		6		1
......	...	...	...	...	...	...	...	...	...	...	...	...	...	...

- The numerators in the *second* slanted row (bold-faced) are the coefficients of the h^{+1} powers which shows that the coefficient of the h^{+1} power in x_0^n is n.
- We check that the *third* slanted row are the coefficients of the h^{+2} powers which shows that the coefficient of the h^{+2} power in x_0^n is $\dfrac{n(n-1)}{2}$.

constant approximation
affine approximation
quadratic approximation

- Etc

17.4 Approximations

Fortunately, most of the time we only need the very first few terms of the addition formulas.

1. Very often, we will need only the **constant approximation** of $(x_0 + h)^n$ which is just x_0^n. Indeed, very often h will be small enough that we will not need to consider any of the monomials that involve it and we will write:

$$(x_0 + h)^n = x_0^n + (...)$$

Example 9. The *constant approximation* of 16.072^7 is 16^7 and we write

$$16.072^7 = 16^7 + (...)$$

More generally, the *constant approximation* of $(16 + h)^7$ is 16^7 and we write

$$(16 + h)^7 = 16^7 + (...)$$

2. When the *constant* approximation is too crude, we will often use the **affine approximation** of $(x_0 + h)^n$ which is $x_0^n + nx_0h$. Indeed, while h may not be small enough not to matter, the other powers, h^2, h^3 etc being smaller than h can often still be ignored and we will then write

$$(x_0 + h)^n = x_0^n + nx_0^{n-1}h + (...)$$

Example 10. The *affine approximation* of 16.072^7 is $16^7 + 7 \cdot 16^6 \cdot 0.072$ and we write

$$16.072^7 = 16^7 + 7 \cdot 16^6 \cdot 0.072 + (...)$$

More generally, the *affine approximation* of $(16 + h)^7$ is $16^7 + 7 \cdot 16^6 \cdot h + (...)$ and we write

$$(16 + h)^7 = 16^7 + 7 \cdot 16^6 \cdot h + (...)$$

3. And finally we will also use the **quadratic approximation** of $(x_0 + h)^n$ which is $x_0^n + nx_0h + \frac{n(n-1)}{2}x_0h^2$ when we will need more precision that the affine approximation will be able to give us and we will then write

$$(x_0 + h)^n = x_0^n + nx_0h + \frac{n(n-1)}{2}x_0h^2 + (...)$$

Example 11. The *quadratic approximation* of 16.072^7 is $16^7 + 7 \cdot 16^6 \cdot 0.072 + 21 \cdot 16^6 \cdot 0.072^2$ and we write

$$16.072^7 = 16^7 + 7 \cdot 16^6 \cdot 0.072 + 21 \cdot 16^6 \cdot 0.072^2 + (...)$$

More generally, the *quadratic approximation* of $(16+h)^7$ is $16^7 + 7 \cdot 16^6 \cdot h + 21 \cdot 16^6 \cdot h^2$ and we write

$$(16 + h)^7 = 16^7 + 7 \cdot 16^6 \cdot h + 21 \cdot 16^6 \cdot h^2 + (...)$$