assign
round
share
leftovers

Chapter 18

Polynomials 4: Division
(In Descending & Ascending Powers)

We now turn to the last one of the four operation with polynomials: division. However, in order to understand the procedure, we must first take a look at the division procedure in ARITHMETIC.

18.1 Division In Arithmetic

We first look at the *real-world process* and then we look at the corresponding *paper-world procedure.*

1. In the real world, we often encounter situations in which we have to **assign** (equally) the items in a first collection to the items of another collection.

The *process* is to make **rounds** during each of which we *assign* one item of the first collection to each one of the items in the second collection. The process comes to an end when, after a round has been completed,

- there are items left unassigned but not enough to complete another round. The **share** is then the collection of items from the first collection that have been assigned to each item of the second collection and the **leftovers** are the collection of items from the first collection left unassigned after the process has come to an end.

  EXAMPLE 1. In the real world, say we have a collection of seven dollar-bills which we want to assign to each and every person in a collection of three person. We want to know how many dollar-bills we will assign to each person and how many dollar-bills will be left-over.

  i. We make a *first round* during which we hand-out one dollar-bill to each and every person in the collection. This uses three dollar-bills and leaves us with four

division
dividend
divisor
quotient
remainder

dollar-bills after the first round.
ii. We make a *second round*, we hand-out one dollar-bill to each and every person in the collection. This uses another three dollar-bills and leaves us with one dollar-bill after the second round.
iii. If we try to make a *third round*, we find that we cannot complete the third round.
So, the *share* is two dollar-bills and the *leftovers* is one dollar-bill.

or,

- there is no item left unassigned. The *share* is again the collection of items from the first collection that have been assigned to each item of the second collection and there are no *leftovers*.

  Example 2. In the real world, say we have a collection of eight dollar-bills which we want to assign to each and every person in a collection of four person. We want to know how many dollar-bills we will assign to each person and how many dollar-bills will be left-over.
  i. We make a *first round* during which we hand-out one dollar-bill to each and every person in the collection. This uses four dollar-bills and leaves us with four dollar-bills after the first round.
  ii. We make a *second round*, we hand-out one dollar-bill to each and every person in the collection. This uses another four dollar-bills and leaves us with no dollar-bill after the second round.
  iii. So, we cannot make a *third round*.
  So, the *share* is two dollar-bills and there are no leftovers.

2. The paper *procedure* that corresponds to the real-world process is called **division**. *Division* will involve the following language:

- The number-phrase that represents the first collection, that is the collections of items *to be assigned* to the items of the second collection, is called the **dividend**,
- The number-phrase that represents the second collection, that is the collection of items *to which* the items of the first collection are to be assigned, is called the **divisor**,
- The number-phrase that represents the *share* is called the **quotient**,
- The number-phrase that represents the *leftovers* is called the **remainder**.

Example 3. Given a real-world situation with a collection of eight dollar-bills to be assigned to each and every person in a collection of four persons,

- The *dividend* is 7 **Dollars**
- The *divisior* is 3 **Persons**
- The *share* is $2\frac{\textbf{Dollars}}{\textbf{Person}}$
- The *remainder* is 1 **Dollar**

18.2 Elementary School Procedure

trial and error
try
partial product
partial remainder

The *division procedure* taught in elementary schools is a **trial and error** procedure which follows the real-world process closely inasmuch as each *round* is represented by a **try** in which:

i. We use the *multiplication procedure* to find the **partial product** which represents how many items *have been used* by the end of the corresponding *real-world round*.

ii. We use the *subtraction procedure* to find the **partial remainder** which represents how many items, if any, are *left over* by the end of the corresponding *real-world round*.

EXAMPLE 4. In order to divide 987 by 321, we go through the following *tries*:

First try:

i. We multiply the *divisor* 321 by 1 which gives the *partial product* 321:

$$\begin{array}{r} 1 \\ 321\ \overline{)\ 987} \\ 321 \end{array}$$

ii. We subtract the *partial product* 321 from the *dividend* 987 which leaves the *partial remainder* 666:

$$\begin{array}{r} 1 \\ 321\ \overline{)\ 987} \\ \underline{321} \\ 666 \end{array}$$

Second try:

i. We multiply the *divisor* 321 by 2 which gives the *partial product* 642:

$$\begin{array}{r} 2 \\ 321\ \overline{)\ 987} \\ 642 \end{array}$$

ii. We subtract the *partial product* 642 from the *dividend* 987 which leaves the *partial remainder* 345:

$$\begin{array}{r} 2 \\ 321\ \overline{)\ 987} \\ \underline{642} \\ 345 \end{array}$$

Third try:

i. We multiply the *divisor* 321 by 3 which gives the *partial product* 963:

$$\begin{array}{r} 3 \\ 321\ \overline{)\ 987} \\ 963 \end{array}$$

ii. We subtract the *partial product* 963 from the *dividend* 987 which leaves the *partial remainder* 24:

rank
table multiplier
table product

$$\begin{array}{r}
3 \\
321\;\overline{)\;987} \\
963 \\
\hline
24
\end{array}$$

Fourth try:

i. We multiply the *divisor* 321 by 4 which gives the *partial product* 1284:

$$\begin{array}{r}
4 \\
321\;\overline{)\;987} \\
1284
\end{array}$$

ii. We cannot subtract the *partial product* 1284 from the *dividend* 987:

$$\begin{array}{r}
4 \\
321\;\overline{)\;987} \\
1284 \\
\hline
\end{array}$$

Since we cannot complete the fourth try, we go back to the last complete try, that is the third try, and we get that the *quotient* is 3 and the *remainder* 24.

This procedure, though, has two severe shortcomings:

- All these *full multiplications* require a lot of work.
- This procedure will not extend to *polynomials*

18.3 Efficient Division Procedure

We now present a much more efficient procedure that, instead of *full multiplications* to find the digits of the quotient, uses only a *multiplication table*[1] and which, for us, has the further advantages that it extends easily to *polynomials*.

1. By the **rank** of a multiplication table, we will mean the *numerator* common to all the multiplications in that multiplication table. The **table multipliers** correspond to the successive lines in the multiplication table and therefore always range from 1 to 9. The **table products** are the results of the successive multiplications in the multiplication table.

EXAMPLE 5. In the following multiplication table

[1]Educologists will surely claim that this procedure is way beyond the feeble mind of their students. Yet, it seems to be the one taught in most of the world and the procedure that uses "full multiplication" seems to be taught mostly, if not only, in the U.S..

7	× 1	=	7
7	× 2	=	14
7	× 3	=	21
7	× 4	=	28
7	× 5	=	35
7	× 6	=	42
7	× 7	=	49
7	× 8	=	56
7	× 9	=	63

cycles
step
stop
continue

- the *rank* is 7,
- the *table multipliers* range from 1 to 9 (as in all multiplication tables),
- the *table products* range from 7 to 63.

2. The *procedure* consists of successive **cycles**. During each of these *cycles*, we go through the following four **step**s:

Step I. We find a *single digit* of the *quotient* by *trial and error* using only the *multiplication table* whose *rank* is the *first digit* of the *divisor*.

Step II. We find the *partial product* by multiplying the *full divisor* by the *single digit* of the quotient we found in Step I.

Step III. We find the *partial remainder* by subtracting the *partial product* we found in Step II from the *full dividend*.

Step IV. We decide whether we want to:

- **stop** the division,
- **continue** the division.

Example 6. We want to compute

$$\frac{9974.}{312.}$$

so we need to divide 312. *into* 9974., that is

$$312.\ \overline{)\ 9974.}$$

Since the first digit in the *divisor* is 3, we will use the multiplication table of rank 3:

3×1	$=$	3
3×2	$=$	6
3×3	$=$	9
3×4	$=$	12
3×5	$=$	15
3×6	$=$	18
3×7	$=$	21
3×8	$=$	24
3×9	$=$	27

CYCLE 1. We look for the *first* digit of the *quotient*.

Step I. We divide by *trial and error* the *first* digit in the *divisor*, 312., into the *first* digit of the *dividend*, 9974.

Trial 1. We try the *table multiplier* 1

i. When we multiply the *first digit* of the *divisor*, 3, by the *table multiplier* 1 we get the *table product* 3:

312) 9974
3

ii. We subtract the *table product* 3 from the *first digit* of the *dividend*, 9974., which leaves the *remainder* 6:

312.) 9974.
3
6

Trial 2. We try the *table multiplier* 2

i. When we multiply the *first digit* of the *divisor*, 3, by the *table multiplier* 2 we get the *table product* 6:

321) 9974
6

ii. We subtract the *table product* 6 from the *first digit* of the *dividend*, 9974., which leaves the *remainder* 3

312.) 9974.
6
3

Trial 3. We try the *table multiplier* 3

i. When we multiply the *first digit* of the *divisor*, 3, by the *table multiplier*

3 we get the *table product* 9:

$$\begin{array}{r} 312.\ \overline{)\ 9974.} \\ 9 \end{array}$$

ii. We subtract the *table product* 9 from the *first digit* of the *dividend*, 9974., which leaves the *remainder* 0

$$\begin{array}{r} 312.\ \overline{)\ 9974.} \\ 9 \\ \hline 0 \end{array}$$

Trial 4. We try the *table multiplier* 4

i. When we multiply the *first digit* of the *divisor*, 3, by the *table multiplier* 4 we get the *table product* 12:

$$\begin{array}{r} 312.\ \overline{)\ 9974.} \\ 12 \end{array}$$

ii. We cannot subtract the *table product* 12 from the *first digit* of the *dividend*, 9974. .

$$\begin{array}{r} 312.\ \overline{)\ 9974.} \\ 12 \\ \hline \end{array}$$

Since we cannot complete Trial 4, we must go back to the last complete trial, that is Trial 3, from which we get that:

The *first digit of the quotient* will be 3 unless the resulting partial product exceeds the *dividend*.

Step II. We multiply the *full divisor*, 312., by the *first* digit in the quotient, 3:

$$\begin{array}{r} 3 \\ 312.\ \overline{)\ 9974.} \\ 936 \end{array}$$

The *first partial product* is 936 **Tens**.

Step III. We subtract the *first partial product*, 936 **Tens**, from the dividend 9974.:

$$\begin{array}{r} 3 \\ 312.\ \overline{)\ 9974.} \\ 936 \\ \hline 614 \end{array}$$

The *first remainder* is 614. and the first digit in the quotient is 3.

Step IV. We decide if we want to *stop* or to *continue* the division:

- If we decide to *stop* the division,
  - the *quotient* of the division is 30. since the first digit of the quotient, 3, refers to the **Tens** and the only denominator that goes without saying is the **Ones**.

– the *remainder* of the division is 614.

If we don't care about the *remainder*, we write:

$$\frac{9974}{312} = 30 + (...)$$

where we write $+ (...)$ as a reminder that $\frac{9974}{312}$ is not exactly equal to 30 since there was a *remainder*.

- If we decide to *continue* the division,

  i. we recall that the 3 in the quotient refers to the **Tens**

$$\begin{array}{rl} & \;\;3 \\ \cline{2-2} 312 & \big)\, 9974 \\ & \;\;936 \\ \cline{2-2} & \;\;\;614 \end{array}$$

  ii. we recall that the remainder is 614 **Ones**,

$$\begin{array}{rl} & \;\;3. \\ \cline{2-2} 312 & \big)\, 9974 \\ & \;\;936 \\ \cline{2-2} & \;\;\;614 \end{array}$$

  iii. we start a new cycle.

CYCLE 2. We look for the *second* digit of the *quotient*.

Step I. We divide by *trial and error* the *first* digit in the *divisor*, 312., into the *first* digit of the *first remainder*, 614. :

Trial 1. We try the *table multiplier* 1

i. When we multiply the *first digit* of the *divisor*, 3, by the *table multiplier* 1 we get the *table product* 3:

$$\begin{array}{rl} & \;\;3 \\ \cline{2-2} 312. & \big)\, 9974. \\ & \;\;936 \\ \cline{2-2} & \;\;\;614 \\ & \;\;\;3 \end{array}$$

ii. We subtract the *table product* 3 from the *first digit* of the *first remainder*, 614., which leaves the *remainder* 3:

$$\begin{array}{rl} & \;\;3 \\ \cline{2-2} 312. & \big)\, 9974. \\ & \;\;936 \\ \cline{2-2} & \;\;\;614 \\ & \;\;\;3 \\ \cline{2-2} & \;\;\;6 \end{array}$$

Trial 2. We try the *table multiplier* 2

i. When we multiply the *first digit* of the *divisor*, 3, by the *table multiplier* 2 we get the *table product* 6:

```
         3
312. ) 9974.
       936
        614
         6
```

ii. We subtract the *table product* 6 from the *first digit* of the *first remainder*, 614., which leaves the *remainder* 0:

```
         3
312. ) 9974.
       936
        614
        6
        0
```

Trial 3. We don't need to do Trial 3 since we obviously will not be able to subtract the *table product* from the *first remainder*.
The second digit of the quotient will be 2 unless the resulting partial product exceeds the *first remainder*.

Step II. We multiply the *full divisor*, 312., by the *second* digit in the quotient, 2:

```
        32
312 ) 9974
      936
       614
       624
```

The *second partial product* is 624 **Ones**

Step III. We cannot subtract the *second partial product*, 624 from the *first remainder*, 614:

```
       3.2
312 ) 9974
      936
       614
       624
```

What happened here is due to the carryover in the multiplication.
So, the *second* digit in the quotient is the *table multiplier* in Trial 1, 1, and we must redo **Step II** and **Step III**:

New **Step II.** We multiply the *full divisor*, 312., by the *second* digit in the quotient, 1:

```
         31
312. ) 9974.
       936
        614
        312
```

The *second partial product* is 312 **Ones**

New **Step III.** We subtract the *second partial product*, 312 **Ones**, from the *first remainder* 614. :

```
          31
312. ) 9974.
        936
        614.
         312
         302
```

The *second remainder* is 302. and the second digit in the quotient is 1.

Step IV. We decide if we want to *stop* or *continue* the division.

- If we decide to *stop* the division,
  - the *quotient* of the division is 31. since the second digit of the quotient, 1, refers to the **Ones**.
  - the *remainder* of the division is 302.

  If we don't care about the *remainder*, we write:

  $$\frac{9974.}{312.} = 31. + (...)$$

  where we write $+ (...)$ as a reminder that $\frac{9974.}{312.}$ is not exactly equal to $31.$ since there was a *remainder*.
- If we decide to *continue* the division,
  i. we point the 1 in the quotient to indicate that it refers to the **Ones**

  ```
            31.
  312. ) 9974.
          936
          614
  ```

  ii. we change the remainder 302 **Ones** to 3020 **Tenths**

  ```
            31.
  312. ) 9974.
          936
          614
          312
          3020
  ```

  iii. we start a new cycle.

CYCLE 3. We look for the *third* digit of the *quotient*.

Step I. We divide by *trial and error* the *first* digit in the *divisor*, 312., into the *first two* digits of the *second remainder*, 3020. :

Trial 1. We try the *table multiplier* 1

i. When we multiply the *first digit* of the *divisor*, 3, by the *table multiplier* 1 we get the *table product* 3:

```
            31.
312. ) 9974.
        936
        ---
         614
         312
         ----
         3020
         3
```

ii. We subtract the *table product* 3 from the *first two digits* of the *second remainder*, 3020., which leaves the *remainder* 27:

```
            31.
312. ) 9974.
        936
        ---
         614
         312
         ----
         3020
          3
         --
         27
```

Trial 2. We try the *table multiplier* 9

i. When we multiply the *first digit* of the *divisor*, 3, by the *table multiplier* 9 we get the *table product* 27:

```
            31.
312. ) 9974.
        936
        ---
         614
         312
         ----
         3020
         27
```

ii. We subtract the *table product* 27 from the *first two digits* of the *second remainder*, 3020., which leaves the *remainder* 3:

```
            31.
312. ) 9974.
        936
        ---
         614
         312
         ----
         3020
         27
         --
          3
```

The third digit of the quotient will be 9 unless the resulting partial product exceeds the third remainder.

Step II. We multiply the *full divisor*, 312., by the *third* digit in the quotient, 9:

$$\begin{array}{rr} & 31.9 \\ \cline{2-2} 312.\ \big) & 9974. \\ & 936 \\ \cline{2-2} & 614 \\ & 312 \\ \cline{2-2} & 3020 \\ & 2808 \end{array}$$

The *third partial product* is 2808 **Tenths**

Step III. We subtract the *third partial product*, 2808 from the *second remainder*, 3020:

$$\begin{array}{rr} & 31.9 \\ \cline{2-2} 312.\ \big) & 9974. \\ & 936 \\ \cline{2-2} & 614 \\ & 312 \\ \cline{2-2} & 3020 \\ & 2808 \\ \cline{2-2} & 212 \end{array}$$

The *third remainder* is 31.2 and the third digit of the quotient is 9

Step IV. We decide if we want to *stop* or *continue* the division.

- If we decide to *stop* the division,
  - the *quotient* of the division is 31.9 since the third digit of the quotient, 9, refers to the **Tenths**.
  - the *remainder* of the division is 21.2

  If we don't care about the *remainder*, we write:

  $$\frac{9974.}{312.} = 31.9 + (...)$$

  where we write $+ (...)$ as a reminder that $\frac{9974.}{312.}$ is not exactly equal to 31.9 since there was a *remainder*.
- If we decide to *continue* the division,
  i. we recall that the 9 in the quotient refers to the **Tenths**

  $$\begin{array}{rr} & 31.9 \\ \cline{2-2} 312.\ \big) & 9974. \\ & 936 \\ \cline{2-2} & 614 \\ & 312 \\ \cline{2-2} & 3020 \\ & 2808 \\ \cline{2-2} & 212 \end{array}$$

  ii. we change the remainder 212 **Tenths** to 2120 **Hundredths**

```
           31.9
312. ) 9974.
       936
        614
        312
        3020
        2808
         2120
```

iii. we start a new cycle.

3. While this procedure certainly appears to be a lot more complicated than the elementary school procedure, it isn't really and it just requires getting used to and taking the time to get used to it is a good investment because, in the long run, this procedure is much more economical since:

- We find the digits of the quotient using only one *multiplication table*,
- We then usually need only do one full multiplication and one subtraction (per cycle) as opposed to one for each try.
- We can decide exactly where we want to stop and see how precise the quotient then would be.

18.4 Division of Polynomials

Since *decimal numerators* are combinations of powers of TEN, it should not be surprising that the above procedure should work for *polynomials* which are combinations of powers of x.

The *procedure* consists of successive *cycles*, one for each monomial in the quotient. During each of these *cycles*, we go through four *step*s:

Step I. We find each *monomial* of the *quotient* by dividing the *first monomial* in the divisor into the *first monomial* of the previous partial remainder.

Step II. We find the *partial product* by multiplying the *full divisor* by the *monomial* of the quotient we found in Step I.

Step III. We find the *partial remainder* by subtracting the *partial product* we found in Step II from the previous partial remainder or, if there is not yet a partial remainder, from the *full dividend*.

Step IV. We decide if we

- *stop* the division
- *continue* the division

Just as, in ARITHMETIC, we can stop the division anywhere we want and we need not stop a division when the quotient reaches a monomial with exponent 0 because we can always divide a monomial into another since we can have *negative* exponents. In fact, again just as in ARITHMETIC, there are cases where we absolutely need to go beyond the exponent 0 and use negative exponents. (See Epilogue.)

EXAMPLE 7. In order to compute $\dfrac{-12x^3+11x^2-17x+1}{-3x^2+5-2}$, we divide $-3x^2+5-2$ into $-12x^3+11x^2-17x+1$:

$$-3x^2 \; +5x \; -2 \;\big)\overline{\; -12x^3 \quad +11x^2 \quad -17x \quad +1}$$

we proceed as follows:

CYCLE 1. Step I. We find the *first monomial in the quotient* by dividing the *first monomial in the divisor*, $-3x^2$, into the *first monomial of the dividend*, $-12x^3$ that is $\dfrac{-12x^3}{-3x^2} = +4x$

$$\begin{array}{rrrrrr} & & +4x & & & \\ -3x^2 & +5x \; -2 \;\big) & -12x^3 & +11x^2 & -17x & +1 \end{array}$$

Step II. We find the *first partial product* by multiplying the *full divisor* by the *first monomial in the quotient*:

$$\begin{array}{rrrrr} & +4x & & & \\ -3x^2 \; +5x \; -2 \;\big) & -12x^3 & +11x^2 & -17x & +1 \\ \text{First partial product:} & -12x^3 & +20x^2 & -8x & \end{array}$$

Step III. We find the *first partial remainder* by *subtracting* the first partial product from the full dividend:

$$\begin{array}{rrrrr} & +4x & & & \\ -3x^2 \; +5x \; -2 \;\big) & -12x^3 & +11x^2 & -17x & +1 \\ \ominus & -12x^3 & +20x^2 & -8x & \end{array}$$

But to *subtract* the first partial product means to *add the opposite* of the first partial product to the full dividend:

$$\begin{array}{rrrrr} & +4x & & & \\ -3x^2 \; +5x \; -2 \;\big) & -12x^3 & +11x^2 & -17x & +1 \\ \oplus & +12x^3 & -20x^2 & +8x & \\ \hline \text{First remainder:} & +0x^3 & -9x^2 & -9x & +1 \end{array}$$

Step IV. We decide if we want to *stop* or *continue* the division.

- If we decide to *stop* the division,
  - the *quotient* of the division is $+4x$.
  - the *remainder* of the division is $-9x^2-8x+1$

  If we don't care about the *remainder*, we write:

$$\frac{-12x^3+11x^2-17x+1}{-3x^2+5x-2} = +4x + (...)$$

  where we write $+(...)$ as a reminder that $\dfrac{-12x^3+11x^2-17x+1}{-3x^2+5x-2}$ is not exactly equal to $+4x$ since there was a *remainder*.

- If we decide to *continue* the division, we start a new cycle

CYCLE 2. Step I. We find the *second monomial in the quotient* by dividing the *first monomial in the divisor*, $-3x^2$, into *the first monomial in the first partial remainder*, $-9x^2$, that is $\frac{-9x^2}{-3x^2} = +3$

$$\begin{array}{r|rrrr} & +4x & +3 & & \\ \hline -3x^2 \;+5x\; -2 & -12x^3 & +11x^2 & -17x & +1 \\ & -12x^3 & +20x^2 & -8x & \\ \hline & & -9x^2 & -9x & +1 \end{array}$$

Step II. We find the *second partial product* by multiplying the *full divisor* by the *second monomial in the quotient*:

$$\begin{array}{r|rrrr} & +4x & +3 & & \\ \hline -3x^2 \;+5x\; -2 & -12x^3 & +11x^2 & -17x & +1 \\ & -12x^3 & +20x^2 & -8x & \\ \hline & & -9x^2 & -9x & +1 \\ \text{Second partial product:} & & -9x^2 & +15x & -6 \end{array}$$

Step III. We find the *second partial remainder* by *subtracting* the second partial product from the first partial remainder:

$$\begin{array}{r|rrrr} & +4x & +3 & & \\ \hline -3x^2 \;+5x\; -2 & -12x^3 & +11x^2 & -17x & +1 \\ & -12x^3 & +20x^2 & -8x & \\ \hline & & -9x^2 & -9x & +1 \\ & \ominus & -9x^2 & +15x & -6 \\ \hline \end{array}$$

But to *subtract* the second partial product means to *add the opposite* of the second partial product to the first partial remainder:

$$\begin{array}{r|rrrr} & +4x & +3 & & \\ \hline -3x^2 \;+5x\; -2 & -12x^3 & +11x^2 & -17x & +1 \\ & -12x^3 & +20x^2 & -8x & \\ \hline & & -9x^2 & -9x & +1 \\ & \oplus & +9x^2 & -15x & +6 \\ \hline \text{Second remainder:} & & +0x^2 & -24x & +7 \end{array}$$

Step IV. We decide if we want to *stop* or *continue* the division.

- If we decide to *stop* the division,
  - the *quotient* of the division is $+4x+3$.
  - the *remainder* of the division is $-24x+7$

  If we don't care about the *remainder*, we write:

$$\frac{-12x^3+11x^2-17x+1}{-3x^2+5x-2} = +4x+3+(\ldots)$$

where we write $+\,(\ldots)$ as a reminder that $\frac{-12x^3+11x^2-17x+1}{-3x^2+5x-2}$ is not exactly equal to $+4x+3$ since there was a *remainder*.

- If we decide to *continue* the division, we start a new cycle

CYCLE 3. Step I. We find the *third monomial in the quotient* by dividing the *first monomial in the divisor*, $-3x^2$, into *the first monomial in the second partial remainder*, $-24x$ that is $\dfrac{-24x}{-3x^2} = +8x^{-1}$

$$\begin{array}{rrrl|rrrrr}
 & & & & +4x & +3 & +8x^{-1} & & \\ \hline
-3x^2 & +5x & -2 &) & -12x^3 & +11x^2 & -17x & +1 & \\
 & & & & -12x^3 & +20x^2 & -8x & & \\ \hline
 & & & & & -9x^2 & -9x & +1 & \\
 & & & & & +9x^2 & -15x & +6 & \\ \hline
 & & & & & & -24x & +7 &
\end{array}$$

Step II. We find the *third partial product* by multiplying the *full divisor* by the *third monomial in the quotient*:

$$\begin{array}{rrrl|rrrrr}
 & & & & +4x & +3 & +8x^{-1} & & \\ \hline
-3x^2 & +5x & -2 &) & -12x^3 & +11x^2 & -17x & +1 & \\
 & & & & -12x^3 & +20x^2 & -8x & & \\ \hline
 & & & & & -9x^2 & -9x & +1 & \\
 & & & & & +9x^2 & -15x & +6 & \\ \hline
 & & & & & & -24x & +7 & \\
\text{Third partial product:} & & & & & & -24x & +40 & -16x^{-1}
\end{array}$$

Step III. We find the *third partial remainder* by *subtracting* the third partial product from the first partial remainder:

$$\begin{array}{rrrl|rrrrr}
 & & & & +4x & +3 & +8x^{-1} & & \\ \hline
-3x^2 & +5x & -2 &) & -12x^3 & +11x^2 & -17x & +1 & \\
 & & & & -12x^3 & +20x^2 & -8x & & \\ \hline
 & & & & & -9x^2 & -9x & +1 & \\
 & & & & & +9x^2 & -15x & +6 & \\ \hline
 & & & & & & -24x & +7 & \\
 & & & & \ominus & & -24x & +40 & -16x^{-1}
\end{array}$$

But to *subtract* the second partial product means to *add the opposite* of the second partial product to the first partial remainder:

$$\begin{array}{rrrl|rrrrr}
 & & & & +4x & +3 & +8x^{-1} & & \\ \hline
-3x^2 & +5x & -2 &) & -12x^3 & +11x^2 & -17x & +1 & \\
 & & & & -12x^3 & +20x^2 & -8x & & \\ \hline
 & & & & & -9x^2 & -9x & +1 & \\
 & & & & & +9x^2 & -15x & +6 & \\ \hline
 & & & & & & -24x & +7 & \\
 & & & & \oplus & & +24x & -40 & +16x^{-1} \\ \hline
\text{Third remainder:} & & & & & & 0x & -33 & +16x^{-1}
\end{array}$$

Step IV. We decide if we want to *stop* or *continue* the division.

- If we decide to *stop* the division,

– the *quotient* of the division is $+4x+3+8x^{-1}$.

– the *remainder* of the division is $-33+16x^{-1}$

If we don't care about the *remainder*, we write:

$$\frac{-12x^3+11x^2-17x+1}{-3x^2+5x-2} = +4x+3+8x^{-1}+(\ldots)$$

where we write $+(\ldots)$ as a reminder that $\dfrac{-12x^3+11x^2-17x+1}{-3x^2+5x-2}$ is not exactly equal to $+4x+3+8x^{-1}$ since there was a *remainder*.

- If we decide to *continue* the division, we start a new cycle

The procedure to divide polynomials is in fact a lot simpler than the procedure for dividing in ARITHMETIC:

- There is never any "carryover"
- The first term of each partial remainder has coefficient 0
- There are no Trials in **Step I** because, when we divide the first monomial in the divisor into the first monomial of a partial remainder, we always get a coefficient for the corresponding monomial in the quotient and the worst that can happen is that this coefficient is a fraction.

EXAMPLE 8. In order to divide $2x^3+5x^2-6$ by $3x-1$ we write (in the *anglo-saxon* tradition):

$$\begin{array}{r|lllll}
 & & \frac{2}{3}x^2 & +\frac{17}{9}x & +\frac{17}{27} \\ \hline
3x-1 & 2x^3 & +5x^2 & & -6 \\
 & -2x^3 & +\frac{2}{3}x^2 & & \\ \cline{2-3}
 & & \frac{17}{3}x^2 & & \\
 & & -\frac{17}{3}x^2 & +\frac{17}{9}x & \\ \cline{3-4}
 & & & \frac{17}{9}x & -6 \\
 & & & -\frac{17}{9}x & +\frac{17}{27} \\ \cline{4-5}
 & & & & -\frac{145}{27}
\end{array}$$

The *quotient* is

$$+\frac{2}{3}x^2+\frac{17}{9}x+\frac{17}{27}$$

The *remainder* is

$$-\frac{145}{27}$$

18.5 Default Rules for Division

Since mathematicians are lazy,

- mathematicians do not write the $+$ sign in front of the coefficients of leading monomials,
- mathematicians do not write monomials with 0 coefficient,

and, most dangerously,

- mathematicians want to write only one stage in Step III but there are two traditions concerning what then to write, as a result, in Step II:
  - In the *latin* tradition, in Step II, we write the *partial product*, that is what we get it from the *multiplication*, and so in Step III, when it comes to subtracting, we visualize the *opposite of the partial product* we wrote in Step II and we oplus what we *visualize*. The advantage is that each line is exactly what we get from the previous operation.

    EXAMPLE 9.

$$\begin{array}{r|llll} & +4x & +3 & & \\ \hline -3x^2 \ +5x \ -2 & -12x^3 & +11x^2 & -16x & +1 \\ & -12x^3 & +20x^2 & -8x & \\ \hline & & -9x^2 & -8x & +1 \end{array}$$

  - In the *anglo-saxon* tradition, we anticipate the subtraction to be done in Step III and in Step II *we write the opposite of the partial product* so in Step III we oplus what we *wrote* in Step II.

    EXAMPLE 10.

$$\begin{array}{r|llll} & +4x & +3 & & \\ \hline -3x^2 \ +5x \ -2 & -12x^3 & +11x^2 & -16x & +1 \\ & +12x^3 & -20x^2 & +8x & \\ \hline & & -9x^2 & -8x & +1 \end{array}$$

From now on we will of course follow the *anglo-saxon* tradition.

EXAMPLE 11. In order to compute $\dfrac{6x^3 + 13x^2 + 13x + 7}{2x + 1}$, we divide $2x + 1$ into $6x^3 + 13x^2 + 13x + 7$:

CYCLE 1. Step I. We find the *first monomial* in the quotient by *short division*:

$$\begin{array}{r|l} & 3x^2 \\ \hline 2x+1 & 6x^3 + 13x^2 + 13x + 7 \end{array}$$

Step II. We get the *first opposite product* by writing the opposite of the result of the *full multiplication*

$$\begin{array}{r|l} & 3x^2 \\ \hline 2x+1 & 6x^3 + 13x^2 + 13x + 7 \\ & \underline{-6x^3 \ - 3x^2} \end{array}$$

Step III. We get the *first remainder* by oplussing the first opposite product

$$\begin{array}{r|l} & 3x^2 \\ \hline 2x+1 & 6x^3 + 13x^2 + 13x + 7 \\ & \underline{-6x^3 \ - 3x^2} \\ & \qquad 10x^2 + 13x \end{array}$$

Step IV. We decide if we want to stop or continue the division
– If we decide to *stop* the division,

∗ the *quotient* of the division is $+3x^{+2}$.

∗ the *remainder* of the division is $+10x^{+2} + 13x$

If we don't care about the *remainder*, we write:

$$\frac{6x^3 + 13x^2 + 13x + 7}{2x + 1} = +3x^{+2} + (\ldots)$$

where we write $+\ (\ldots)$ as a reminder that $\dfrac{6x^3 + 13x^2 + 13x + 7}{2x + 1}$ is not exactly equal to $+3x^{+2}$ since there was a *remainder*.

– If we decide to *continue* the division, we start a new cycle

CYCLE 2. Step I. We find the *second monomial* in the quotient by *short division*:

$$\begin{array}{r|l} & \quad 3x^2 \quad + 5x \\ \hline 2x+1) & \quad 6x^3 + 13x^2 + 13x + 7 \\ & -6x^3 \quad -3x^2 \\ \hline & \qquad\quad 10x^2 + 13x \end{array}$$

Step II. We get the *second opposite product* by writing the opposite of the result of the *full multiplication*

$$\begin{array}{r|l} & \quad 3x^2 \quad + 5x \\ \hline 2x+1) & \quad 6x^3 + 13x^2 + 13x + 7 \\ & -6x^3 \quad -3x^2 \\ \hline & \qquad\quad 10x^2 + 13x \\ & \qquad -10x^2 \quad -5x \\ \hline \end{array}$$

Step III. We get the *second remainder* by oplussing the fir second st opposite product

$$\begin{array}{r|l} & \quad 3x^2 \quad + 5x \\ \hline 2x+1) & \quad 6x^3 + 13x^2 + 13x + 7 \\ & -6x^3 \quad -3x^2 \\ \hline & \qquad\quad 10x^2 + 13x \\ & \qquad -10x^2 \quad -5x \\ \hline & \qquad\qquad\qquad 8x + 7 \end{array}$$

Step IV. We decide if we want to stop or continue the division

– If we decide to *stop* the division,

∗ the *quotient* of the division is $+3x^{+2} + 5x$.

∗ the *remainder* of the division is $+8x + 7$

If we don't care about the *remainder*, we write:

$$\frac{6x^3 + 13x^2 + 13x + 7}{2x + 1} = +3x^{+2} + 5x + (\ldots)$$

where we write $+\ (\ldots)$ as a reminder that $\dfrac{6x^3 + 13x^2 + 13x + 7}{2x + 1}$ is not exactly equal to $+3x^{+2} + 5x$ since there was a *remainder*.

– If we decide to *continue* the division, we start a new cycle

CYCLE 3. Step I. We find the *third monomial* in the quotient by *short division*:

$$\begin{array}{r|l} & 3x^2 \quad + 5x + 4 \\ \hline 2x+1 & 6x^3 + 13x^2 + 13x + 7 \\ & -\,6x^3 \quad - 3x^2 \\ \hline & 10x^2 + 13x \\ & -\,10x^2 \quad - 5x \\ \hline & 8x + 7 \end{array}$$

Step II. We get the *third opposite product* by writing the opposite of the result of the *full multiplication*

$$\begin{array}{r|l} & 3x^2 \quad + 5x + 4 \\ \hline 2x+1 & 6x^3 + 13x^2 + 13x + 7 \\ & -\,6x^3 \quad - 3x^2 \\ \hline & 10x^2 + 13x \\ & -\,10x^2 \quad - 5x \\ \hline & 8x + 7 \\ & -\,8x - 4 \\ \hline \end{array}$$

Step III. We get the *third remainder* by oplussing the third opposite product

$$\begin{array}{r|l} & 3x^2 \quad + 5x + 4 \\ \hline 2x+1 & 6x^3 + 13x^2 + 13x + 7 \\ & -\,6x^3 \quad - 3x^2 \\ \hline & 10x^2 + 13x \\ & -\,10x^2 \quad - 5x \\ \hline & 8x + 7 \\ & -\,8x - 4 \\ \hline & 3 \end{array}$$

Step IV. We decide if we want to stop or continue the division

– If we decide to *stop* the division,

  ∗ the *quotient* of the division is $+3x^{+2} + 5x + 4$.

  ∗ the *remainder* of the division is $+3$

  If we don't care about the *remainder*, we write:

$$\frac{6x^3 + 13x^2 + 13x + 7}{2x + 1} = +3x^{+2} + 5x + 4 + (...)$$

  where we write $+\,(...)$ as a reminder that $\dfrac{6x^3 + 13x^2 + 13x + 7}{2x + 1}$ is not exactly equal to $+3x^{+2} + 5x + 4$ since there was a *remainder*.

– If we decide to *continue* the division, we start a new cycle

- When writing the partial remainders, we do not write the monomials beyond those that result from subtracting the *partial product*.

EXAMPLE 12.

	$+4x$	$+3$		
$-3x^2\ +5x\ -2$)	$-12x^3$	$+11x^2$	$-16x$	$+1$
First *opposite* partial product:	$+12x^3$	$-20x^2$	$+8x$	
First remainder:		$-9x^2$	$-8x$	
Second *opposite* partial product:		$+9x^2$	$-15x$	$+6$
Second remainder:			$-23x$	$+7$

The danger here is that, when we do the next subtraction, we may subtract from 0 rather than from the monomial that was unwritten in the partial remainder.

18.6 Division in Ascending Powers

One major difference between ARITHMETIC and POLYNOMIAL ALGEBRA is that:

- In ARITHMETIC, the *base* in the powers is always *larger* than ONE—in our case it is TEN but, for instance, COMPUTER SCIENCES use TWO, EIGHT and SIXTEEN as well.
- In POLYNOMIAL ALGEBRA, the base in the powers can be *smaller* than ONE as well as *larger* than ONE and, while this has no effect on the procedures we use for *addition*, *subtraction* and *multiplication*, whether x stands for numbers larger than 1 or smaller than 1 makes all the difference in the case of *division*.

This is because division usually does not stop by itself and *we* have to decide when to stop it. But we want to make sure that, after we have replaced the *unspecified numerator* by a *specific numerator*, what we kept of the quotient will give us *most* of what we should get so that we want the size of the successive results to go *diminishing*.

Now, as we already mentioned in Chapter 15, Section 2,

- When x is to be replaced by a numerator that is going to be *large in size*, we will want the Laurent polynomial to be written in *descending order of exponents*.
- When x is to be replaced by a numerator that is going to be *small in size*, we will want the Laurent polynomial to be written in *ascending order of exponents*.

As a result, we need to be able to divide in *ascending* order of exponents as well as in *descending* order of exponents. Fortunately, the procedure is exactly the same.

EXAMPLE 13. In order to compute $\dfrac{-12+23h-h^2-2h^3}{-3+2h}$, we divide $-3+2h$ into $-12+23h-h^2-2h^3$:

	$+4$	$-5h$	$-3h^2$	
$-3\ +2h$)	-12	$+23h$	$-h^2$	$-2h^3$
First *opposite* partial product:	$+12$	$-8h$		
First remainder:		$+15h$	$-h^2$	$-2h^3$
Second *opposite* partial product:		$-15h$	$+10h^2$	
Second remainder:			$+9h^2$	$-2h^3$
Third *opposite* partial product:			$-9h^2$	$+6h^3$
Third remainder:				$+4h^3$

- If we decide to *stop* the division,
  - the *quotient* of the division is $\boxed{+4-5h-3h^2}$.
  - the *remainder* of the division is $\boxed{+4h^3}$. Observe that if we replace the unspecified numerator h by, say, 0.2, then the remainder is equal to $4 \bullet 0.2^3 = 4 \bullet 0.008 = 0.032$ which is indeed small.

  If we don't care about the *remainder*, we write:

  $$\boxed{\frac{-12+23h-h^2-2h^3}{-3+2h} = +4-5h-3h^2+(\ldots)}$$

  where we write $+\ (\ldots)$ as a reminder that $\dfrac{-12+23h-h^2-2h^3}{-3+2h}$ is not exactly equal to $+4-5h-3h^2$ since there was a *remainder*.
- If we decide to *continue* the division, we start a new cycle