Epilogue

The reader may have been curious as to what Part II - Inequations and Equations and Part III - Laurent Polynomials could have in common or why, of all the topics in algebra, these two were chosen here.

Moreover, in the last two chapters, a new idea made its first appearance, namely that there are at least two cases when we *approximate* the result of a procedure:

- One case is when we compute the powers of a binomial, we can conceivably do the whole computation but the point was that a lot of the work involved to get the exact result would really be wasted and that it would turn out that we would be perfectly happy with only an *approximation* of the result.
- The other case is when we divide and there it is not a case where we could do the whole computation because the division needs not terminate by itself. Fortunately, since the *remainder* keeps get smaller and smaller, a point has to come, sooner or later, where the precision in the quotient would become unnecessary and so we can terminate and stay with an *approximation* of the *quotient*.

But, while we illustrated the idea of *approximation* in the case of arithmetic, we didn't show where and how the idea of approximation would come in algebra.

1. Functions

As usual, we need to build things up a bit before we can get to the actual point.

In the real world, things are always changing, if sometimes very slowly, and we will begin with is to see how we represent this idea on paper.

1. To perceive that something is *changing*, it is necessary to look at it against something else that either does *not* change at all or that changes *differently*. For instance, the amount of income tax changes *in terms of*

input
output
function
input numerator
output numerator
input-output table
input-output rule
unspecified input

income, the amount of property tax changes *in terms of* assets, the amount of sales tax changes *in terms of* expenses.

More precisely, in order to observe something changing, we must pair each of the stages that it goes through in terms of the stages that some reference thing goes through, if only a clock or a calendar!

EXAMPLE 14. We might say that, in 2003, someone's income tax was $6,753. Just to have said that the income tax was $6,753 would not say much since $6,753 was a lot less money in, say, 2007 than it was in 1913 when income tax was first created.

The reference stages will be called **inputs** and the stages of what we are investigating will be called **outputs**.

We will call **function** the *pairing* of **input numerators** with **output numerators** that results from any process, device, procedure, agency, converter, exchanger, translator, etc that satisfies the condition that an input cannot result in more than one output. This, by the way, does not mean that two inputs cannot result in the same output: they can.

EXAMPLE 15. A *parking meter* is a function because, given an input, say 1 **Quarter**, the parking meter returns a definite amount of parking time, say 30 **Minutes**. We would then say that (1 **Quarter**, 30 **Minutes**) is an input-output pair.

EXAMPLE 16. A *slot machine* is not a function because, given an input, say 1 **Quarter**, a slot machine could return *any* number of **Quarters**.

2. While in some sciences, such as PSYCHOLOGY and SOCIOLOGY, *functions* are usually specified by **input-output tables**, in other sciences, such as PHYSICS and ELECTRONICS, *functions* are usually specified by **input-output rules** as follows:

i. We use a letter, usually x, as **unspecified input**. In other words, we will be able to replace x by any *specific input* we want.

ii. We must either have or give a name to the function. In the absence of any given name, we shall usually use the letter f.

iii. Then, $f(x)$ will stand for the *output* returned for the input x by the function f.

EXAMPLE 17. If a function called, say, FUN doubles the input and adds 5 to give the output, then the *input-output rule* of FUN is:

$$x \xrightarrow{FUN} FUN(x) = 2x + 5$$

Then, the output for a *specific* input, say 7, is

$$2x + 5|_{x:=7}$$

So, in order to compute the output for the given input 7, we replace all the occurrences of x in the input-output rule by 7 and then we compute

rational function

$$\begin{aligned} x := 7 \xrightarrow{FUN} FUN(x)|_{x:=7} &= 2x+5|_{x:=7} \\ &= 2\cdot 7+5 \\ &= 14+5 \\ &= 19 \end{aligned}$$

We can then write

$$7 \xrightarrow{FUN} FUN(7) = 19$$

3. A particular type of function, called **rational function**, is when the output is in the form of a fraction where both the top and the bottom are *polynomials*.

Example 18. The function RAT whose input-output rule is

$$x \xrightarrow{RAT} RAT(x) = \frac{3x^2+5x-4}{x^3-8}$$

is a *rational* function.
When the input is, say, 3, we compute the output as follows:

$$\begin{aligned} x := 3 \xrightarrow{RAT} RAT(x)|_{x:=3} &= \left.\frac{3x^2+5x-4}{x^3-8}\right|_{x:=3} \\ &= \frac{3\cdot(3)^2+5\cdot(3)-4}{(3)^3-8} \\ &= \frac{27+15-4}{27-8} \\ &= \frac{38}{19} \\ &= 2 \end{aligned}$$

4. We shall now look at two kinds of problems that we encounter in the investigation of functions:

- *Local* investigations in which the main tools are those of Part III - Laurent Polynomials
- *Global* investigations in which the main tools are those of Part II - Inequations and Equations

and, in fact, if perhaps surprisingly, *local* and *global* problems have almost nothing in common in the sense that usually no amount of local investigation will help in a global problem and, vice versa, no global investigation will shed much light on a local problem.

2. Local Problems

While it is usually relatively easy to compute the *output* of a given function for a given *input*, this is usually not what we are after because knowing what the output is for a given input may say nothing about the output for a second input even when the second input is very close to the first input.

1. Most of the time, indeed, two inputs that are *close* will give outputs that are also *close*.

EXAMPLE 19. Given the function RAT whose input-output rule is

$$x \xrightarrow{RAT} RAT(x) = \frac{3x^2+5x-4}{x^3-8}$$

we saw in EXAMPLE 5 that the input 3 gives the output 2 and we would like now to get the output when the input is *near* 3, say when it is 3.1.

$$\begin{aligned} x := 3+0.1 \xrightarrow{RAT} RAT(x)\big|_{x:=3+0.1} &= \left.\frac{3x^2+5x-4}{x^3-8}\right|_{x:=3+0.1} \\ &= \frac{3\cdot(3+0.1)^2+5\cdot(3+0.1)-4}{(3+0.1)^3-8} \\ &= \frac{3\cdot[3^2+(...)]+5\cdot[3+(...)]-4}{[3^3+(...)]-8} \\ &= \frac{27+(...)+15+(...)-4+(...)}{27+(...)-8} \\ &= \frac{38+(...)}{19+(...)} \\ &= 2+(...) \end{aligned}$$

So, we have that

$$3.1 \xrightarrow{RAT} RAT(3.1) = 2+(...)$$

In other words, the input 3.1 which is close to the input 3 gives an output that is close to the output of 3.

2. Quite often, though, it can happen that two inputs that are *close* will give outputs that are *far apart*.

EXAMPLE 20. Given again the function RAT whose input-output rule is

$$x \xrightarrow{RAT} RAT(x) = \frac{3x^2+5x-4}{x^3-8}$$

we will now show that, while the inputs 1.9 and 2.1 are *close*, their outputs are *far apart*.

In order to save time and energy, we will compute the output for $2+h$ and only at the end we will replace h by -0.1 on the one hand and by $+0.1$ on the other hand.

$$x := 2 + h \xrightarrow{RAT} RAT(x)|_{x:=2+h} = \left.\frac{3x^2 + 5x - 4}{x^3 - 8}\right|_{x:=2+h}$$
$$= \frac{3 \cdot (2+h)^2 + 5 \cdot (2+h) - 4}{(2+h)^3 - 8}$$
$$= \frac{3 \cdot [2^2 + 2 \cdot 2 \cdot h + (...)] + 5 \cdot [2+h] - 4}{[2^3 + 3 \cdot 2^2 \cdot h + (...)] - 8}$$
$$= \frac{12 + 12h + (...) + 10 + 5h - 4}{8 + 12h + (...) - 8}$$
$$= \frac{18 + 17h + (...)}{12h + (...)}$$

and the division in *ascending exponents* gives

$$= \frac{18}{12} h^{-1} + (...)$$

Now:

- when we replace h by -0.1, we get that

$$x := 2 - 0.1 \xrightarrow{RAT} RAT(x)|_{x:=2-0.1} = \frac{18}{12} \cdot (-0.1)^{-1} + (...)$$
$$= \frac{18}{12} \cdot (-10) + (...)$$
$$= -\frac{180}{12} + (...)$$
$$= -15 + (...)$$

- while when we replace h by $+0.1$, we get that

$$x := 2 + 0.1 \xrightarrow{RAT} RAT(x)|_{x:=2+0.1} = \frac{18}{12} \cdot (+0.1)^{-1} + (...)$$
$$= \frac{18}{12} \cdot (+10) + (...)$$
$$= +\frac{180}{12} + (...)$$
$$= +15 + (...)$$

so that, even though the *inputs* were only 0.2 apart, their *outputs* are $30 + (...)$ apart.

3. While we may tend to expect functions to give *large* outputs only for *large* inputs, this is far from being necessarily the case. In fact, most rational functions do not behave that way at all and

- *large* inputs can give *non-large* outputs,

zero
pole

EXAMPLE 21. Given the function TIT whose input-output rule is

$$x \xrightarrow{\quad TIT \quad} TIT(x) = \frac{3x+2}{x^3+5}$$

we compute its output when the input is $1,000$:

$$\begin{aligned} x := 1,000 \xrightarrow{\quad TIT \quad} TIT(x)|_{x:=1,000} &= \left.\frac{3x+2}{x^3+5}\right|_{x:=1,000} \\ &= \frac{3 \cdot 1,000 + (...)}{1,000^3 + (...)} \\ &= \frac{3,000 + (...)}{1,000,000,000 + (...)} \\ &= \frac{3}{1,000,000} + (...) \\ &= 0.000003 + (...) \end{aligned}$$

which is certainly *non-large*.

- *non-large* inputs can give *large* outputs.

EXAMPLE 22. Given the function TAT whose input-output rule is

$$x \xrightarrow{\quad TAT \quad} TAT(x) = \frac{x^2+3}{x-7}$$

we compute its output when the input is 7.01. In fact, we compute the output when the input is $7+h$ and we let $h := 0.01$ only at the end:

$$\begin{aligned} x := 7+h \xrightarrow{\quad TAT \quad} TAT(x)|_{x:=7+h} &= \left.\frac{x^2+3}{x-7}\right|_{x:=7+h} \\ &= \frac{(7+h)^2+3}{(7+h)-7} \\ &= \frac{[7^2+(...)]+3}{h} \\ &= \frac{7^2+3+(...)}{h} \\ &= \frac{52}{h} + (...) \end{aligned}$$

and when we replace h by 0.01, we get that the output is $5,200$ which is certainly *large*.

An input near which the outputs are *small* is called a **zero** because the output for a *zero* is 0. By symmetry, an input near which the outputs are *large* is called a **pole** and we will say that the output for a *pole* is ∞.

4. More generally, given inputs that are either:

- near and on either side of a non-large input x_0

or

- near and on either side of ∞, that is that are large *in size*

local investigations can be about finding:

- the sign of the *slope*, that is whether the graph of the function is *going* UP or *going* DOWN,
- the sign of the *bending*, that is whether the graph of the function is *bending* UP or *bending* DOWN,
- whether the output is the LARGEST or the SMALLEST as compared to the outputs for neighboring inputs.

3. Global Problems

Global problems are those where we are looking for *input*(s), if any, whose *output* has a required feature.

1. Just as with what was already the case in PART II - INEQUATIONS AND EQUATIONS, the *zeros*, that is the inputs whose output is 0 play an important role in the investigation of functions. But, in the case of rational functions, the *pole*(s), that is the inputs whose output is ∞, also play an important role.

EXAMPLE 23. Given the rational function HOM whose input-output rule is:

$$x \xrightarrow{\quad HOM \quad} HOM(x) = \frac{3x-15}{7x+14}$$

find the *input*(*s*), if any, whose *output* is *positive*.

In other words, we need to solve the *rational problem* in which the *data set* consists of all signed decimal numerators

$$\frac{3x-15}{7x+14} > 0$$

which we do essentially in the same manner as in PART II - INEQUATIONS AND EQUATIONS, that is we use the PASCH PROCEDURE:

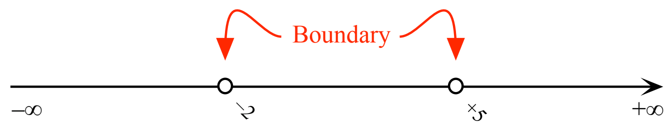
I. We determine the *boundary* of the solution subset. These are

- the solution(s), if any, of the associated equation $\dfrac{3x-15}{7x+14} = 0$, that is the *zero*(*s*), if any, of the function HOM, that is the solution(s), if any, of the equation
$$3x - 15 = 0$$
- the solution(s), if any, of the associated equation $\dfrac{3x-15}{7x+14} = \infty$, that is the *pole*(*s*), if any, of the function HOM, that is the solution(s), if any, of the equation
$$7x + 14 = 0$$

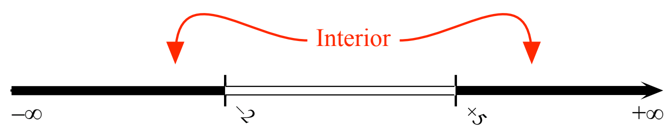
In other words, the *boundary* is the solution subset of the associated double problem

$$\text{OR}\begin{cases} 3x - 15 = 0 \\ 7x + 14 = 0 \end{cases}$$

Proceeding as in Chapter 12, we get that the graph of the *boundary* is:

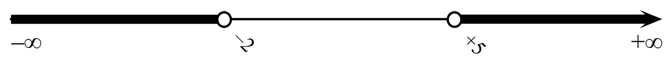


II. We determine the *interior* of the solution subset by testing each one of the three sections separated by the boundary points and then using the PASCH THEOREM. We get that the graph of the *interior* is



III. Altogether, the *inputs* whose *output* by the function HOM is *positive* are represented by:

- The *graph* of the solution subset is (we use DEFAULT RULE #4)



- The *name* of the solution subset is

$$\left(-\infty, -2\right) \cup \left(+5, +\infty\right)$$

2. More generally, *global* investigations can be about finding:

- all those *input*(s), if any, for which the *slope* of the local graph is equal to 0,
- all those *input*(s), if any, for which the local graph *goes* UP (or *goes* DOWN),
- all those *input*(s), if any, for which the *bending* of the local graph is equal to 0,
- all those *input*(s), if any, for which the local graph *bends* UP (or *bends* DOWN),
- all those *input*(s), if any, whose output is LARGER (or SMALLER) than the output of all neighboring inputs.

4. Conclusion

While its purpose was to show both how PART II - INEQUATIONS AND EQUATIONS and PART III - LAURENT POLYNOMIALS were fundamental tools in the investigation of *functions* and how unavoidable, but also how powerful a tool, was the idea of *approximation*, this Epilogue couldn't really do justice to a very rich subject called the DIFFERENTIAL CALCULUS and, beyond that, to its extension called INITIAL VALUE PROBLEMS.

We are however unable to resist ending this book with an example of an *initial value problem.*

EXAMPLE 24. Imagine a pond with an inexhaustible amount of weeds in which there are two *populations* of fish:

- Herbivorous fish, that is fish that feed on the weeds,
- Carnivorous fish, that is fish that feed on the herbivorous fish.

We would like to follow these two populations as time goes by.

Suppose we know what the two populations are at the beginning of time, for instance that there are many more herbivorous fish than carnivorous fish. Then the population of carnivorous fish is going to go UP. But, as the population of carnivorous fish goes UP, they eat more and more of the herbivorous fish whose population is going to go DOWN. But then, so will the population of carnivorous fish. Etc.

What seems to be critical here are the relative rates at which the two populations of fish reproduce and grow and, from that knowledge, one should be able to figure out what the the two populations are going to be at any time.

On paper, one represents each one of the two populations by a function whose input is time and whose output is the number of fish. One then tries to write equations that represent the real-world situation just described and, in fact, this representation of the real-world situation is called the Lotka-Volterra's double differential equation problem after the two people who first wrote and investigated, independently of each other, these equations.

Hopefully, then, this Epilogue will turn out to be only a Prologue to a thorough investigation of FUNCTIONS, a concept central not only to MATHEMATICS but to many other scientific subjects as well.