Preface

The prospect facing students still in need of Basic Algebra as they enter two-year colleges[2] is a discouraging one inasmuch as it usually takes at the very least two semesters before they can arrive at the course(s) that they are interested in—or required to take, not to dwell on the fact that their chances of *overall* success tend to be extremely low[3].

Reasonable Basic Algebra (RBA) is a standalone version of part of From Arithmetic To Differential Calculus (A2DC), a *course of study* developed to allow a significantly higher percentage of students to complete Differential Calculus in three semesters. As it is intended for a one-semester course, though, RBA may serve in a similar manner students with different goals.

The general intention is to get the students to change from being "answer oriented", the inevitable result of "show and tell, drill and test", to being "question oriented[4]" and thus, rather than try to "remember" things, be able to "reconstruct" them as needed. The specific means by which RBA hopes to accomplish this goal are presented at some length below but, briefly, they include:

- An *expositional approach*, based on what is known in mathematics as model theory, which carefully distinguishes "real-world" situations from their "paper-world" representations[5]. A bit more precisely, we start with processes involving "real-world" collections that yield either a relationship between these collections or some new collection and the students then have to develop a paper procedure that will yield the sentence representing the relationship or the number-phrase representing the new

[2]Otherwise known, these days, as "developmental" students.

[3]For instance, students who wish eventually to learn Differential Calculus, the "mathematics of change", face five or six semesters with chances of overall success of no more than one percent.

[4]See John Holt's classic *How Chidren Fail*, Delacorte Press,1982.

[5]See Zoltan P. Dienes, for instance *Building Up Mathematics*.

collection.

EXAMPLE 1. Given that, in the real-world, when we attach to a collection of three apples to a collection of two apples we get a collection of five apples, the question for the students is to develop a paper procedure that, from 3 **Apples** and 2 **Apples**, the number-phrases representing on paper these real-world collections, will yield the number-phrase 5 **Apples**.

In other words, the students are meant to abstract the necessary concepts from a familiar "real-world" since, indeed, "We are usually more easily convinced by reasons we have found ourselves than by those which have occurred to others." (Blaise Pascal).

- A very carefully structured *contents architecture*—in total contrast to the usual more or less haphazard string of "topics"—to create systematic reinforcement and foster an exponential learning curve based on a Coherent View of Mathematics and thus help students acquire a Profound Understanding of Fundamental Mathematics[6].
- A systematic attention to *linguistic issues* that often prevent students from being able to focus on the *mathematical concepts* themselves.
- An insistence on *convincing* the students that the reason things mathematical are the way they are is not because "experts say so" but because *common sense* says they *cannot* be otherwise.

$\therefore$

The *contents architecture* was designed in terms of three major requirements.

1. From the *students*' viewpoint, each and every mathematical issue should:

- flow "naturally" from what just precedes it,
- be developed only as far as *needed* for what will follow "naturally",
- be dealt with in sufficient "natural" *generality* to support further developments without having first to be recast.

EXAMPLE 2. After counting dollars sitting on a counter, it is "natural" to count dollars changing hands over the counter and thus to develop signed numbers. In contrast, multiplication, division or fractions all involve a complete change of venue.

2. Only a very few very simple but very powerful *ideas* should be used to underpin all the presentations and discussions even if this may be at the cost of some additional length. After they have *familiarized* themselves with such an idea, in its simplest possible embodiment, later, in more complicated situations, the students can then focus on the *technical* aspects of getting

[6] See Liping Ma's *Knowing and Teaching Elementary School Mathematics.*

the idea to *work* in the situation at hand. In this manner, the students eventually get to feel that they can *cope* with "anything".

EXAMPLE 3. The concept of *combination-phrase* is introduced with 3 **Quarters** + 7 **Dimes** in which **Quarters** and **Dimes** are *denominators* and where + does *not* denote addition as it does in 3 **Quarters** + 7 **Quarters** but stands for "and". (In fact, for a while, we write 3 **Quarters** & 7 **Dimes**.) The concept then comes up again and again: with 3 HUNDREDS + 7 TENS, with $\frac{3}{4} + \frac{7}{10}$, with $3x^2 + 7x^5$, with $3x + 7y$, etc, culminating, if much later, with $3\vec{\imath} + 7\vec{\jmath}$.

EXAMPLE 4. *If* we can *change*, say, 1 **Quarter** for 5 **Nickels** and 1 **Dime** for 2 **Nickels**, we can then change the *combination-phrase* 3 **Quarters** + 7 **Dimes** for 3 ~~Quarters~~ × $\frac{\text{5 Nickels}}{\text{Quarter}}$ + 7 ~~Dimes~~ × $\frac{\text{2 Nickels}}{\text{Dime}}$ that is for the *specifying-phrase* 15 **Nickels** + 14 **Nickels** which we *identify* as 29 **Nickels**. (Note by the way that here × is a very particular type of multiplication, as also found in 3 ~~Dollars~~ × $\frac{\text{7 Cents}}{\text{Dollar}}$ = 21 **Cents**.) Later, when having to "add" $\frac{3}{4} + \frac{7}{10}$, the students will then need only to concentrate on the *technical* issue of developing a procedure to find the denominators that **Fourth** and **Tenth** can *both* be changed for, e.g. **Twentieths**, **Hundredths**, etc.

3. The issue of "undoing" whatever has been done should always be, if not always resolved, at least always discussed.

EXAMPLE 5. Counting *backward* is introduced by the need to undo counting *forward* and both *subtracting* and *signed* numbers are introduced by the need to *undo* adding, that is by the need to solve the *equation* $a + x = b$.

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As a result of these requirements, the *contents* had to be stripped of the various "kitchen sinks" to be found in current BASIC ALGEBRA courses and the two essential themes RBA focuses on are *affine inequations & equations* and *Laurent polynomials*. This focus *empowers* the students in that, once they have mastered these subjects, they will be able both: **i.** to investigate the CALCULUS OF FUNCTIONS as in A2DC and **ii.** to acquire in a similar manner whatever other algebraic tools they may need for other purposes.

However, a problem arose in that the background necessary for a treatment that would make solid sense to the students was not likely to have been acquired in any course the students might have taken previously while, for lack of time, a full treatment of ARITHMETIC, such as can be found in A2DC, was out of the question here.

Following is the "three PARTS compromise" that was eventually reached. PART I consists of a treatment of ARITHMETIC, taken from A2DC but minimal in two respects: i. It is limited to what is strictly necessary to make sense of *inequations* & *equations* in Part II and *Laurent polynomials* in Part III, that is to the ways in which number-phrases are *compared* and *operated* *with.* ii. It is developed only in the case of *counting* number-phrases with the extension to *decimal* number-phrases to be taken for granted even though the latter are really of primary importance—and fully dealt with in A2DC.

- Chapter 1 introduces and discusses the general model theoretic concepts that are at the very core of RBA: *real-world collections* versus *paper-world number-phrases*, *combinations*, *graphic representations*.
- Chapter 2 discusses *comparisons*, with real-world collections compared *cardinally*, that is by way of one-to-one matching, while paper-world number-phrases are compared *ordinally*, that is by way of counting. The six *verbs*, $<$, $>$, $\leqq$, $\geqq$, $=$, $\neq$, together with their interrelationships, are carefully discussed in the context of *sentences*, namely *inequalities* and *equalities* that can be TRUE or FALSE.
- Chapter 3 discusses the *effect* of an *action* on a *state* and introduces *addition* as a *unary* operator representing the real-world *action* of attaching a collection to a collection.
- Chapter 4 introduces *subtraction* as a *unary* operator meant to "undo" addition, that is as representing the real-world *action* of detaching a collection from a collection.
- Chapter 5 considers collections of "two-way" items which we represent by *signed number-phrases*.

  EXAMPLE 6. Collections of steps forward versus collections of steps backward, Collections of steps up versus collections of steps down, Collections of dollars gained versus collections of dollars lost, etc

  In order to deal with *signed* number-phrases, the *verbs*, $<$, $>$, etc, are extended to $\circledless$, $\circledgtr$, etc and the *operators* $+$ and $-$ to $\oplus$ and $\ominus$.
- Chapter 6 introduces *co-multiplication* between number-phrases and *unit-value* number-phrases as a way to find the *value* that represents the *worth* of a collection.

  EXAMPLE 7. 3 **Apples** $\times\, 2\frac{\textsf{Cents}}{\textsf{Apple}} = 6$ **Cents** as well as 3 **Dollars** $\times\, 7\frac{\textsf{Cents}}{\textsf{Dollar}} = 21$ **Cents**

  We continue to distinguish between *plain* number-phrases and *signed* number-phrases with $\times$ and $\otimes$.

PART II then deals with number-phrases *specified* as solution of *problems*.

- Chapter 7 introduces the idea of real-world collections selected from a set of selectable collections by a requirement and, in the paper-world, of nouns specified from a data set by a form. Letting the data set then consist of *counting* numerators, we discuss *locating* and *representing* the solution subset (of the data set) specified by a *basic* formula, i.e. of type $x = x_0$, $x < x_0$, etc where x_0 is a given *gauge*.
- Chapter 8 extends the previous ideas to the case of *decimal* numerators by introducing a general procedure, to be systematically used henceforth, in

which we locate separately the *boundary* and the *interior* of the solution subset. Particular attention is given to the representation of the solution subset, both by *graph* and by *name*.

- Chapter 9 begins the focus on the computations necessary to *locate* the boundary in the particular case of "special affine" problems, namely *translation* problems and *dilation* problems, which are solved by *reducing* them to *basic* problems.
- Chapter 10 then solves *affine* problems by *reducing* them to *dilation* problems and hence to *basic* problems. It concludes with the consideration of some *affine-reducible* problems.
- Chapter 11 discusses the *connectors* AND, AND/OR, EITHER/OR, in the context of *double* basic problems, that is problems involving two *basic* inequations/equations (in the same unknown). Here again, particular attention is given to the representation of the solution subset, both by *graph* and by *name*.
- Chapter 12 wraps up the discussion of how to select collections with the investigation of *double* affine problems, that is problems involving two affine inequations/equations (in the same unknown).

PART III investigates *plain polynomials* as a particular case of *Laurent polynomials*.

- Chapter 13 discusses what is involved in *repeated multiplications* and *repeated divisions* of a number-phrase by a *numerator* and introduces the notion of *signed* power.
- Chapter 14 extends this notion to *Laurent monomials*, namely signed powers of x. Multiplication and division or *Laurent monomials* are carefully discussed.
- Chapter 15 extends the fact that *decimal* numerators are *combinations* of signed powers of TEN to the introduction of *Laurent* polynomials as combinations of *signed* powers of x. Addition and subtraction of polynomials are then defined in the obvious manner.
- Chapter 16 continues the investigation of Laurent polynomials with the investigation of multiplication.
- Chapter 17 discusses a particular case of multiplication, namely the successive powers of $x_0 + u$.
- Chapter 18 closes the book with a discussion of the division of polynomials both in descending and ascending powers

$\therefore$

This is probably the place where it should be disclosed that, as the development of this text was coming to an end, the author came across a 1905 text[7] that gave him the impression that, in his many deviations from the current praxis, he had often reinvented the wheel. While rather reassuring, this was also, if perhaps surprisingly, somewhat disheartening.

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Some of the *linguistic issues* affecting the students's progress are very specific and are directly addressed *as such.* The concept of *duality*, for instance, is a very powerful one and occurs in very many guises.

- When it occurs as "passive voice", *duality* is almost invariably confused with *symmetry*, a more familiar concept[8]. But, in particular, while duality preserves *truth*, symmetry may or may not.

  EXAMPLE 8. "**Jack** is a *child* of **Sue**" is the *dual* of "**Sue** is a *parent* of **Jack**" and, since both refer to the same real-world relationship, they are either both TRUE or both FALSE.
  On the other hand, "**Jack** is a *child* of **Sue**" is the *symmetrical* of "**Sue** is a *child* of **Jack**" and, *here*, the truth of one forces the falsehood of the other. But compare with what would happen with "brother" or "sibling" instead of "child".

- When it occurs as *indirect* definition, *duality* is quite foreign to most students but absolutely indispensable in certain situations.

  EXAMPLE 9. While **Dollar** can be defined *directly* in terms of **Quarters** by saying that 1 **Dollar** is equal to 4 **Quarters**, the definition of **Quarter** in terms of **Dollar** is an *indirect* one in that we must say that a **Quarter** is *that* kind of coin *of which* we need 4 to change for 1 **Dollar** and students first need to be reconciled with this syntactic form. The same stumbling block occurs in dealing with roots since $\sqrt{9}$ is to be understood as "*that* number the square *of which* is 9"[9].

Other linguistic issues, even though more diffuse, are nevertheless systematically taken into account. For instance:

- While mathematicians are used to all sorts of things "going without saying", students feel more comfortable when everything is made *explicit* as, for instance, when & is distinguished from +. Hence, in particular, the *explicit* use in this text of *default rules.*
- The meaning of mathematical symbols usually depends on the context while students generally feel more comfortable with *context-free* termi-

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[7] H. B. Fine, *College Algebra*, reprinted by American Mathematical Society Chelsea, 2005.

[8] The inability to use the "passive voice" is a most important *linguistic* stumbling block for students and one that Educologists have yet to acknowledge.

[9] Educologists will surely agree that, for instance, these particular "reverse" problems would in fact be better dealt with in an *algebraic* context, i.e. as the investigation of $4x = 1$ and $x^2 = 9$. Iincidentally, this is the point of view adopted in A2DC where arithmetic and algebra are systematically "integrated".

nology, that is in the case of a *one-to-one* correspondence between *terms* and *concepts*.

- Even small linguistic variations in *parallel* cases disturb the students who take these variations as having to be significant and therefore as implying in fact an unsaid but actual lack of parallelism.

In general, being aware of what *needs* to be said versus what can go without saying is part of what makes one a mathematician and, as such, requires learning and getting used to. Thus, although being pedantic is not the goal here, RBA tries very hard to be as pedestrian as possible and, if only for the purpose of "discussing matters", to make sure that *everything* is *named* and that every term is "explained" even if usually not *formally* defined.

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The standard way of establishing truth in *mathematics* is by way of proof but the capacity of being *convinced* by a proof is another part of what makes one a mathematician. And indeed, since the students for whom RBA was written are used only to drill based on "template examples", they tend to behave as in the joke about Socrates' slave who, when led through the proof of the Pythagorean Theorem, answers "Yes" when asked if he agrees with the current step and "No" when asked at the end if he agrees with the truth of the Theorem. So, to try to be *convincing*, we use a mode of *arguing* somewhat like that used by lawyers in front of a *court*[10].

Another reason for using a mode of reasoning more akin to everyday argumentation is that even people unlikely to become prospective mathematicians ought to realize the similarities between having to establish the truth in *mathematics* and having to establish the truth in *real-life*. Yet, as Philip Ross wrote recently, "*American psychologist Edward Thorndike first noted this lack of transference over a century ago, when he showed that* [...] *geometric proofs do not teach the use of logic in daily life.*"[11].

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Finally, it is perhaps worth mentioning that this text came out of the author's conviction that it is not good for a society to have a huge majority of its citizens saying they were "never good in math". To quote Colin McGinn at some length:

"*Democratic States are constitutively committed to ensuring and furthering the intellectual health of the citizens who compose them: indeed, they are*

[10]See Stephen E. Toulmin, *The Uses of Argument* Cambridge University Press, 1958

[11]Philip E. Ross, *The Expert Mind*. Scientific American, August 2006.

only possible at all if people reach a certain cognitive level [...]. *Democracy and education (in the widest sense) are thus as conceptually inseparable as individual rational action and knowledge of the world.* [...] *Plainly,* [education] *involves the transmission of knowledge from teacher to taught. But* [knowledge] *is true justified belief that has been arrived at by rational means.* [...] *Thus the norms governing political action incorporate or embed norms appropriate to rational belief formation.* [...]"

"*A basic requirement is to cultivate in the populace a respect for intellectual values, an intolerance of intellectual vices or shortcomings.* [...] *The forces of cretinisation are, and have always been, the biggest threat to the success of democracy as a way of allocating political power: this is the fundamental conceptual truth, as well as a lamentable fact of history.*"

"[However] *people do not really like the truth; they feel coerced by reason, bullied by fact. In a certain sense, this is not irrational, since a commitment to believe only what is true implies a willingness to detach your beliefs from your desires.* [...] *Truth limits your freedom, in a way, because it reduces your belief-options; it is quite capable of forcing your mind to go against its natural inclination.* [...] *One of the central aims of education, as a preparation for political democracy, should be to enable people to get on better terms with reason—to learn to live with the truth.*" [12]

[12] Colin McGinn, *Homage to Education*, London Review of Books, August 16, 1990