Preface

The prospect facing students still in need of Basic Algebra as they enter two-year colleges is a discouraging one inasmuch as it usually takes at the very least two semesters before they can arrive at the course(s) that they are interested in—or required to take, not to dwell on the fact that their chances of overall success tend to be extremely low.

Reasonable Basic Algebra (RBA) is a standalone version of part of From Arithmetic To Differential Calculus (A2DC), a course of study developed to allow a significantly higher percentage of students to complete Differential Calculus in three semesters. As it is intended for a one-semester course, though, RBA may serve in a similar manner students with different goals.

The general intention is to get the students to change from being “answer oriented”, the inevitable result of “show and tell, drill and test”, to being “question oriented” and thus, rather than try to “remember” things, be able to “reconstruct” them as needed. The specific means by which RBA hopes to accomplish this goal are presented at some length below but, briefly, they include:

• An expositional approach, based on what is known in mathematics as model theory, which carefully distinguishes “real-world” situations from their “paper-world” representations. A bit more precisely, we start with processes involving “real-world” collections that yield either a relationship between these collections or some new collection and the students then have to develop a paper procedure that will yield the sentence representing the relationship or the number-phrase representing the new

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2 Otherwise known, these days, as “developmental” students.
3 For instance, students who wish eventually to learn Differential Calculus, the “mathematics of change”, face five or six semesters with chances of overall success of no more than one percent.
5 See Zoltan P. Dienes, for instance Building Up Mathematics.
collection.

**Example 1.** Given that, in the real-world, when we attach to a collection of three apples to a collection of two apples we get a collection of five apples, the question for the students is to develop a paper procedure that, from 3 Apples and 2 Apples, the number-phrases representing on paper these real-world collections, will yield the number-phrase 5 Apples.

In other words, the students are meant to abstract the necessary concepts from a familiar “real-world” since, indeed, “We are usually more easily convinced by reasons we have found ourselves than by those which have occurred to others.” (Blaise Pascal).

- A very carefully structured *contents architecture*—in total contrast to the usual more or less haphazard string of “topics”—to create systematic reinforcement and foster an exponential learning curve based on a Coherent View of Mathematics and thus help students acquire a Profound Understanding of Fundamental Mathematics.
- A systematic attention to *linguistic issues* that often prevent students from being able to focus on the *mathematical concepts* themselves.
- An insistence on *convincing* the students that the reason things mathematical are the way they are is not because “experts say so” but because common sense says they *cannot* be otherwise.

The *contents architecture* was designed in terms of three major requirements.

1. From the *students’* viewpoint, each and every mathematical issue should:
   - flow “naturally” from what just precedes it,
   - be developed only as far as *needed* for what will follow “naturally”,
   - be dealt with in sufficient “natural” *generality* to support further developments without having first to be recast.

**Example 2.** After counting dollars sitting on a counter, it is “natural” to count dollars changing hands over the counter and thus to develop signed numbers. In contrast, multiplication, division or fractions all involve a complete change of venue.

2. Only a very few very simple but very powerful *ideas* should be used to underpin all the presentations and discussions even if this may be at the cost of some additional length. After they have *familiarized* themselves with such an idea, in its simplest possible embodiment, later, in more complicated situations, the students can then focus on the *technical* aspects of getting

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6See Liping Ma’s *Knowing and Teaching Elementary School Mathematics*. 
the idea to work in the situation at hand. In this manner, the students eventually get to feel that they can cope with “anything”.

**Example 3.** The concept of combination-phrase is introduced with 3 Quarters + 7 Dimes in which Quarters and Dimes are denominators and where + does not denote addition as it does in 3 Quarters + 7 Quarters but stands for “and”. (In fact, for a while, we write 3 Quarters & 7 Dimes.) The concept then comes up again and again: with 3 Hundreds + 7 Tens, with \( \frac{3}{4} + \frac{7}{10} \), with \( 3x^2 + 7x^5 \), with \( 3x + 7y \), etc, culminating, if much later, with \( 3i + 7j \).

**Example 4.** If we can change, say, 1 Quarter for 5 Nickels and 1 Dime for 2 Nickels, we can then change the combination-phrase 3 Quarters + 7 Dimes for 3 Quarters \( \times \) \( \frac{5}{3} \) Nickels + 7 Dimes \( \times \) \( \frac{2}{1} \) Nickels, that is for the specifying-phrase 15 Nickels + 14 Nickels which we identify as 29 Nickels. (Note by the way that here \( \times \) is a very particular type of multiplication, as also found in 3 Dollars \( \times \) \( \frac{7}{10} \) Cents = 21 Cents.) Later, when having to “add” \( \frac{3}{4} + \frac{7}{10} \), the students will then need only to concentrate on the technical issue of developing a procedure to find the denominators that Fourth and Tenth can both be changed for, e.g. Twentieths, Hundredths, etc.

3. The issue of “undoing” whatever has been done should always be, if not always resolved, at least always discussed.

**Example 5.** Counting backward is introduced by the need to undo counting forward and both subtracting and signed numbers are introduced by the need to undo adding, that is by the need to solve the equation \( a + x = b \).

\[ \therefore \]

As a result of these requirements, the contents had to be stripped of the various “kitchen sinks” to be found in current basic algebra courses and the two essential themes RBA focuses on are affine inequations & equations and Laurent polynomials. This focus empowers the students in that, once they have mastered these subjects, they will be able both: i. to investigate the calculus of functions as in A2DC and ii. to acquire in a similar manner whatever other algebraic tools they may need for other purposes.

However, a problem arose in that the background necessary for a treatment that would make solid sense to the students was not likely to have been acquired in any course the students might have taken previously while, for lack of time, a full treatment of arithmetic, such as can be found in A2DC, was out of the question here.

Following is the “three parts compromise” that was eventually reached. Part I consists of a treatment of arithmetic, taken from A2DC but minimal in two respects: i. It is limited to what is strictly necessary to make sense of inequations & equations in Part II and Laurent polynomials in Part III, that is to the ways in which number-phrases are compared and operated
with. ii. It is developed only in the case of counting number-phrases with the extension to decimal number-phrases to be taken for granted even though the latter are really of primary importance—and fully dealt with in A2DC.

- Chapter 1 introduces and discusses the general model theoretic concepts that are at the very core of RBA: real-world collections versus paper-world number-phrases, combinations, graphic representations.
- Chapter 2 discusses comparisons, with real-world collections compared cardinally, that is by way of one-to-one matching, while paper-world number-phrases are compared ordinally, that is by way of counting. The six verbs, $<$, $>$, $\leq$, $\geq$, $=$, $\neq$, together with their interrelationships, are carefully discussed in the context of sentences, namely inequalities and equalities that can be TRUE or FALSE.
- Chapter 3 discusses the effect of an action on a state and introduces addition as a unary operator representing the real-world action of attaching a collection to a collection.
- Chapter 4 introduces subtraction as a unary operator meant to “undo” addition, that is as representing the real-world action of detaching a collection from a collection.
- Chapter 5 considers collections of “two-way” items which we represent by signed number-phrases.

**Example 6.** Collections of steps forward versus collections of steps backward, collections of steps up versus collections of steps down, collections of dollars gained versus collections of dollars lost, etc. In order to deal with signed number-phrases, the verbs, $<$, $>$, etc, are extended to $\ominus$, $\oslash$, etc and the operators $+$ and $-$ to $\oplus$ and $\ominus$.
- Chapter 6 introduces co-multiplication between number-phrases and unit-value number-phrases as a way to find the value that represents the worth of a collection.

**Example 7.** $3 \text{ Apples} \times 2 \frac{\text{Cents}}{\text{Apple}} = 6 \text{ Cents}$ as well as $3 \text{ Dollars} \times 7 \frac{\text{Cents}}{\text{Dollar}} = 21 \text{ Cents}$ We continue to distinguish between plain number-phrases and signed number-phrases with $\times$ and $\otimes$.

**Part II** then deals with number-phrases specified as solution of problems.
- Chapter 7 introduces the idea of real-world collections selected from a set of selectable collections by a requirement and, in the paper-world, of nouns specified from a data set by a form. Letting the data set then consist of counting numerators, we discuss locating and representing the solution subset (of the data set) specified by a basic formula, i.e. of type $x = x_0$, $x < x_0$, etc where $x_0$ is a given gauge.
- Chapter 8 extends the previous ideas to the case of decimal numerators by introducing a general procedure, to be systematically used henceforth, in
which we locate separately the boundary and the interior of the solution subset. Particular attention is given to the representation of the solution subset, both by graph and by name.

- Chapter 9 begins the focus on the computations necessary to locate the boundary in the particular case of “special affine” problems, namely translation problems and dilation problems, which are solved by reducing them to basic problems.

- Chapter 10 then solves affine problems by reducing them to dilation problems and hence to basic problems. It concludes with the consideration of some affine-reducible problems.

- Chapter 11 discusses the connectors and, and/or, either/or, in the context of double basic problems, that is problems involving two basic inequations/equations (in the same unknown). Here again, particular attention is given to the representation of the solution subset, both by graph and by name.

- Chapter 12 wraps up the discussion of how to select collections with the investigation of double affine problems, that is problems involving two affine inequations/equations (in the same unknown).

**PART III** investigates plain polynomials as a particular case of Laurent polynomials.

- Chapter 13 discusses what is involved in repeated multiplications and repeated divisions of a number-phrase by a numerator and introduces the notion of signed power.

- Chapter 14 extends this notion to Laurent monomials, namely signed powers of x. Multiplication and division or Laurent monomials are carefully discussed.

- Chapter 15 extends the fact that decimal numerators are combinations of signed powers of ten to the introduction of Laurent polynomials as combinations of signed powers of x. Addition and subtraction of polynomials are then defined in the obvious manner.

- Chapter 16 continues the investigation of Laurent polynomials with the investigation of multiplication.

- Chapter 17 discusses a particular case of multiplication, namely the successive powers of $x_0 + u$.

- Chapter 18 closes the book with a discussion of the division of polynomials both in descending and ascending powers

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This is probably the place where it should be disclosed that, as the development of this text was coming to an end, the author came across
a 1905 text\textsuperscript{7} that gave him the impression that, in his many deviations from the current praxis, he had often reinvented the wheel. While rather reassuring, this was also, if perhaps surprisingly, somewhat disheartening.

Some of the linguistic issues affecting the students’s progress are very specific and are directly addressed as such. The concept of duality, for instance, is a very powerful one and occurs in very many guises.

- When it occurs as “passive voice”, duality is almost invariably confused with symmetry, a more familiar concept.\textsuperscript{8} But, in particular, while duality preserves truth, symmetry may or may not.

**EXAMPLE 8.** “Jack is a child of Sue” is the dual of “Sue is a parent of Jack” and, since both refer to the same real-world relationship, they are either both true or both false.

On the other hand, “Jack is a child of Sue” is the symmetrical of “Sue is a child of Jack” and, here, the truth of one forces the falsehood of the other. But compare with what would happen with “brother” or “sibling” instead of “child.”

- When it occurs as indirect definition, duality is quite foreign to most students but absolutely indispensable in certain situations.

**EXAMPLE 9.** While Dollar can be defined directly in terms of Quarters by saying that 1 Dollar is equal to 4 Quarters, the definition of Quarter in terms of Dollar is an indirect one in that we must say that a Quarter is that kind of coin of which we need 4 to change for 1 Dollar and students first need to be reconciled with this syntactic form. The same stumbling block occurs in dealing with roots since $\sqrt{9}$ is to be understood as “that number the square of which is 9”\textsuperscript{9}.

Other linguistic issues, even though more diffuse, are nevertheless systematically taken into account. For instance:

- While mathematicians are used to all sorts of things “going without saying”, students feel more comfortable when everything is made explicit as, for instance, when $\&$ is distinguished from $\cdot$. Hence, in particular, the explicit use in this text of default rules.

- The meaning of mathematical symbols usually depends on the context while students generally feel more comfortable with context-free termi-
ology, that is in the case of a one-to-one correspondence between terms and concepts.

- Even small linguistic variations in parallel cases disturb the students who take these variations as having to be significant and therefore as implying in fact an unsaid but actual lack of parallelism.

In general, being aware of what needs to be said versus what can go without saying is part of what makes one a mathematician and, as such, requires learning and getting used to. Thus, although being pedantic is not the goal here, RBA tries very hard to be as pedestrian as possible and, if only for the purpose of “discussing matters”, to make sure that everything is named and that every term is “explained” even if usually not formally defined.

\[ \therefore \]

The standard way of establishing truth in mathematics is by way of proof but the capacity of being convinced by a proof is another part of what makes one a mathematician. And indeed, since the students for whom RBA was written are used only to drill based on “template examples”, they tend to behave as in the joke about Socrates’ slave who, when led through the proof of the Pythagorean Theorem, answers “Yes” when asked if he agrees with the current step and “No” when asked at the end if he agrees with the truth of the Theorem. So, to try to be convincing, we use a mode of arguing somewhat like that used by lawyers in front of a court\(^\text{10}\).

Another reason for using a mode of reasoning more akin to everyday argumentation is that even people unlikely to become prospective mathematicians ought to realize the similarities between having to establish the truth in mathematics and having to establish the truth in real-life. Yet, as Philip Ross wrote recently, “American psychologist Edward Thorndike first noted this lack of transference over a century ago, when he showed that [...] geometric proofs do not teach the use of logic in daily life.”\(^\text{11}\).

\[ \therefore \]

Finally, it is perhaps worth mentioning that this text came out of the author’s conviction that it is not good for a society to have a huge majority of its citizens saying they were “never good in math”. To quote Colin McGinn at some length:

“Democratic States are constitutively committed to ensuring and furthering the intellectual health of the citizens who compose them: indeed, they are

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\(^{10}\)See Stephen E. Toulmin, *The Uses of Argument* Cambridge University Press, 1958

only possible at all if people reach a certain cognitive level [...]. Democracy and education (in the widest sense) are thus as conceptually inseparable as individual rational action and knowledge of the world. [...] Plainly, [education] involves the transmission of knowledge from teacher to taught. But [knowledge] is true justified belief that has been arrived at by rational means. [...] Thus the norms governing political action incorporate or embed norms appropriate to rational belief formation. [...]"

“A basic requirement is to cultivate in the populace a respect for intellectual values, an intolerance of intellectual vices or shortcomings. [...] The forces of cretinisation are, and have always been, the biggest threat to the success of democracy as a way of allocating political power: this is the fundamental conceptual truth, as well as a lamentable fact of history.”

“However people do not really like the truth; they feel coerced by reason, bullied by fact. In a certain sense, this is not irrational, since a commitment to believe only what is true implies a willingness to detach your beliefs from your desires. [...] Truth limits your freedom, in a way, because it reduces your belief-options; it is quite capable of forcing your mind to go against its natural inclination. [...] One of the central aims of education, as a preparation for political democracy, should be to enable people to get on better terms with reason—to learn to live with the truth.”