Laurent monomials
coefficient
output-specifying code
exponent
power

Chapter 11

Laurent Monomials

Laurent monomials are number-phrases in which a *given numerator*, referred to as the **coefficient**, is *multiplied* or *divided* by a given number of *copies of an* unspecified numerator x. More precisely,

DEFINITION 11.1. **Laurent monomials** are number-phrases of the form

$$\boxed{\text{coefficient}} \quad \underbrace{\boxed{x^{\text{exponent}}}}_{\textbf{specifying code}}$$

where:

- The coefficient can be any numerator.
- The **exponent** in the specifying code x^{exponent} is a *signed counting* numerator that specifies what is to done to the coefficient with the copies of x:
  - The size of the *exponent* specifies *how many copies* of x are to be made. (If the exponent is 0, no copy is to be made and the coefficient is just to be left alone.)
  - The sign of the *exponent* specifies whether the coefficient is:

    To be multiplied by the copies of x ($+$ sign)

    To be divided by the copies of x ($-$ sign)

evaluate
split equality
comments

EXAMPLE 11.1.

- We read $-317.44\,x^{+3}$ as -317.44 multiplied by 3 copies of x,
- We read $+619.52\,x^{-5}$ as $+619.52$ divided by 5 copies of x.

11.1 Evaluation

1. To **evaluate** a given Laurent monomial for a given numerator:
i. We *declare* that x is to be replaced by the given numerator,
ii. We *decode* the specifying code,
iii. We *execute*.

ATTENTION 11.1. Split Equalities. To make the steps of the procedures clear,
i. We will write on the *left* what we want to get,
ii. We will write on separate lines on the *right* the successive stages of the *computation* that gets us what we want.
iii. Between the steps, there will usually be **comments**.

EXAMPLE 11.2. To evaluate the Laurent monomial $-3\,x^{+4}$ for -4.2:
i. We *declare* that x is to be replaced by -4.2:

$$-3\,x^{+4}\Big|_{x\leftarrow -4.2} =$$

ii. We *decode* the specifying code:

$$= -3 \text{ multiplied by } \underbrace{-4.2 \cdot -4.2 \cdot -4.2 \cdot -4.2}_{4 \text{ copies of } -4.2}$$

iii. We *execute*:

$$\begin{aligned} &= -3 \cdot +311.16 + [...] \\ &= -933.50 + [...] \end{aligned}$$

EXAMPLE 11.3. To evaluate the Laurent monomial $+512\,x^{-3}$ for -4:
i. We *declare* that x is to be replaced by -4

$$+512\,x^{-3}\Big|_{x\leftarrow -4} =$$

$\boxdot$

ii. We *decode* the specifying code:

$$= +512 \text{ divided by } 3 \text{ copies of } -4$$

iii. We *execute*:

$$= \frac{+512}{\underbrace{-4 \cdot \; -4 \; \cdot -4}_{3 \text{ copies of } -4}}$$

$$= \frac{+512}{-64}$$

$$= -8$$

2. Most of the time, though, we delay evaluating Laurent monomial as much as possible and, instead, we *compute* with the Laurent monomial themselves as long as possible, that is until there is nothing else to do but *evaluate* the end Laurent monomial. So, in fact and from now on, we will focus on the *operations* with Laurent monomials.

11.2 Multiplication

We will use the symbol $\boxdot$ to code for the multiplication of Laurent Monomials.

1. We can multiply Laurent monomials just using Definition 11.1 on page 1 as in the following examples.

EXAMPLE 11.4. To execute $\left[-17\, x^{+5}\right] \boxdot \left[+11\, x^{+4}\right]$

i. StartIng from the specifying code

$$\left[-17\, x^{+5}\right] \boxdot \left[+11\, x^{+4}\right] =$$

ii. We decode:

$$= \left[-17 \text{ multiplied by } 5 \text{ copies of } x\right] \cdot \left[+11 \text{ multiplied by } 4 \text{ copies of } x\right]$$

iii. We execute:

$$= \left[-17 \cdot \underbrace{x \cdot x \cdot x \cdot x \cdot x}_{5 \text{ copies of } x}\right] \cdot \left[+11 \cdot \underbrace{x \cdot x \cdot x \cdot x}_{4 \text{ copies of } x}\right]$$

$$= \boxed{-17} \cdot \boxed{+11} \boxed{\cdot} \underbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}_{\boxed{5}+\boxed{4} \text{ copies of } x}$$

iv. We recode:

$$= \boxed{-17} \cdot \boxed{+11}\, x^{\boxed{+}\boxed{5+4}}$$

$$= \boxed{-187}\, x^{\boxed{+}\boxed{9}}$$

Example 11.5. To execute $\left[\boxed{-17}\, x^{\boxed{+}\boxed{7}}\right] \boxdot \left[\boxed{+11}\, x^{\boxed{-}\boxed{3}}\right]$

i. StartIng from the specifying code

$$\left[\boxed{-17}\, x^{\boxed{+}\boxed{7}}\right] \boxdot \left[\boxed{+11}\, x^{\boxed{-}\boxed{3}}\right] =$$

ii. We decode:

$$= \left[\boxed{-17}\ \boxed{\text{multiplied by}}\ \boxed{7} \text{ copies of } x\right] \cdot \left[\boxed{+11}\ \boxed{\text{divided by}}\ \boxed{3} \text{ copies of } x\right]$$

iii. We execute

$$= \left[\boxed{-17} \boxed{\cdot} \underbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}_{\boxed{7} \text{ copies of } x}\right] \cdot \left[\frac{\boxed{+11}}{\underbrace{x \cdot x \cdot x}_{\boxed{3} \text{ copies of } x}}\right]$$

$$= \boxed{-17} \cdot \boxed{+11} \frac{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x}$$

$$= \boxed{-17} \cdot \boxed{+11} \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}}$$

$$= \boxed{-17} \cdot \boxed{+11} \frac{x \cdot x \cdot x \cdot x}{+1}$$

$$= \boxed{-17} \cdot \boxed{+11} \boxed{\cdot} \underbrace{x \cdot x \cdot x \cdot x}_{\boxed{7}-\boxed{3} \text{ copies of } x}$$

iv. We recode

$$= \boxed{-17} \cdot \boxed{+11}\, x^{\boxed{+}\boxed{7-3}}$$

$$= \boxed{-187}\, x^{\boxed{+}\boxed{4}}$$

EXAMPLE 11.6. To execute $\left[-17\, x^{-7}\right] \boxdot \left[+11\, x^{+3}\right]$

i. StartIng from the specifying code

$$\left[-17\, x^{-7}\right] \boxdot \left[+11\, x^{+3}\right] =$$

ii. We decode:

$$= \left[-17 \text{ divided by } 7 \text{ copies of } x\right] \cdot \left[+11 \text{ multiplied by } 3 \text{ copies of } x\right]$$

iii. We execute

$$= \left[\frac{-17}{\underbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}_{7 \text{ copies of } x}}\right] \cdot \left[+11 \cdot \underbrace{x \cdot x \cdot x}_{3 \text{ copies of } x}\right]$$

$$= -17 \cdot +11 \frac{x \cdot x \cdot x}{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}$$

$$= -17 \cdot +11 \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x \cdot x}$$

$$= -17 \cdot +11 \frac{+1}{\underbrace{x \cdot x \cdot x \cdot x}_{7 - 3 \text{ copies of } x}}$$

iv. We recode

$$= -17 \cdot +11\, x^{-\,7-3}$$

$$= -187\, x^{-4}$$

2. We saw in the above examples that *multiplying* Laurent monomials using Definition 11.1 on page 1 is long and painful. But since we used + to code for multiplied by and − to code for divided by so that the exponents are signed counting numerators, it so happens that ⊕ automatically takes care of everything in all cases:

> **THEOREM 11.1 Laurent Monomial Multiplication**
> In order to *multiply* two monomials $ax^{\pm m}$ and $bx^{\pm n}$,
> - We *multiply* the coefficients,
> - We *oplus* the exponents.
>
> In other words:
> $$ax^{\pm m} \boxdot bx^{\pm n} = a \cdot b\, x^{\pm m \oplus \pm n}$$

$\boxdiv$

EXAMPLE 11.7. To execute $\left[-17 \cdot x^{+5}\right] \boxdot \left[+11 \cdot x^{+4}\right]$ using Theorem 11.1 Laurent Monomial Multiplication on page 5 we just write:

$$\begin{aligned}\left[-17 \cdot x^{+5}\right] \boxdot \left[+11 \cdot x^{+4}\right] &= \\ &= -17 \cdot +11\, x^{+5 \oplus +4} \\ &= -187\, x^{+9}\end{aligned}$$

which is of course the same result at in Example 11.4 on page 3.

EXAMPLE 11.8. To execute $\left[-17 \cdot x^{+7}\right] \boxdot \left[+11 \cdot x^{-3}\right]$ using Theorem 11.1 Laurent Monomial Multiplication on page 5 we just write:

$$\begin{aligned}\left[-17 \cdot x^{+7}\right] \boxdot \left[+11 \cdot x^{-3}\right] &= \\ &= -17 \cdot +11\, x^{+7 \oplus -3} \\ &= -187\, x^{+4}\end{aligned}$$

which is of course the same result at in Example 11.5 on page 4.

EXAMPLE 11.9. To execute $\left[-17 \cdot x^{-7}\right] \boxdot \left[+11 \cdot x^{+3}\right]$ using Theorem 11.1 Laurent Monomial Multiplication on page 5 we just write:

$$\begin{aligned}\left[-17 \cdot x^{-7}\right] \boxdot \left[+11 \cdot x^{+3}\right] &= \\ &= -17 \cdot +11\, x^{-7 \oplus +3} \\ &= -187\, x^{-4}\end{aligned}$$

which is of course the same result at in Example 11.6 on page 5.

11.3 Division

We will use the symbol $\boxdiv$ to code for the division of Laurent Monomials.

1. We recall a definition from ARITHMETIC:

DEFINITION 11.2. The **reciprocal** of a given numerator is the numerator that multiplies the given numerator to $+1$. In other words, the reciprocal of x is $\frac{+1}{x}$. In particular, the reciprocal of a fraction is the *upside-down* fraction.

reciprocal

EXAMPLE 11.10. The reciprocal of -13.52 is $-0.07+[...]$ because $\frac{+1}{-13.52} = -0.07+[...]$. Indeed, $-13.52 \odot -0.07+[...] = +1+[...]$

EXAMPLE 11.11. The reciprocal of $\frac{-3.45}{+71.04}$ is $\frac{+71.04}{-3.45}$ because $\frac{-3.45}{+71.04} \odot \frac{+71.04}{-3.45} = \frac{\cancel{-3.45} \odot \cancel{+71.04}}{\cancel{+71.04} \odot \cancel{-3.45}} = +1$

2. Just the way $A \ominus B$ is the same as $A \oplus$ opposite of B, we have

THEOREM 11.2 Reciprocal $A \oplus B$ is the same as $A \odot$ reciprocal of B

3. We can *divide* Laurent monomials just using Definition 11.1 on page 1 as in the following examples but with, when the exponent of the second Laurent monomial is *negative*, the help of Theorem 11.2 Reciprocal on page 7 as in Example 11.14 on page 9 and Example 11.15 on page 9.

EXAMPLE 11.12. To execute $\left[-17\, x^{+7}\right] \oplus \left[+11\, x^{+4}\right]$

i. StartIng from the specifying code

$$\left[-17\, x^{+7}\right] \oplus \left[+11\, x^{+4}\right] =$$

ii. We decode:

$$= \left[-17 \text{ multiplied by } 7 \text{ copies of } x\right] \div \left[+11 \text{ multiplied by } 4 \text{ copies of } x\right]$$

iii. We execute

$$= \left[-17 \cdot \underbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}_{7 \text{ copies of } x}\right] \div \left[+11 \cdot \underbrace{x \cdot x \cdot x \cdot x}_{4 \text{ copies of } x}\right]$$

$$= -17 \div +11 \frac{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x \cdot x}$$

$$= -17 \div +11 \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}}$$

$$= -17 \div +11 \frac{x \cdot x \cdot x}{+1}$$

$$= \boxed{-17} \div \boxed{+11} \cdot \underbrace{x \cdot x \cdot x}_{7 - 4 \text{ copies of } x}$$

iv. We recode

$$= \boxed{-17} \div \boxed{+11}\, x^{+\, 7-4}$$
$$= \boxed{-1.54+[...]}\, x^{+\, 3}$$

Example 11.13. To execute $\left[\boxed{-15}\, x^{+\,5}\right] \boxdiv \left[\boxed{+11}\, x^{+\,9}\right]$

i. StartIng from the specifying code

$$\left[\boxed{-15}\, x^{+\,5}\right] \boxdiv \left[\boxed{+11}\, x^{+\,9}\right] =$$

ii. We decode:

$$= \left[\boxed{-15} \text{ multiplied by } \boxed{5} \text{ copies of } x\right] \div \left[\boxed{+11} \text{ multiplied by } \boxed{9} \text{ copies of } x\right]$$

iii. We execute

$$= \left[\boxed{-15} \cdot \underbrace{x \cdot x \cdot x \cdot x \cdot x}_{5 \text{ copies of } x}\right] \div \left[\boxed{+11} \cdot \underbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}_{9 \text{ copies of } x}\right]$$
$$= \boxed{-15} \div \boxed{+11} \frac{x \cdot x \cdot x \cdot x}{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}$$
$$= \boxed{-15} \div \boxed{+11} \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x \cdot x \cdot x}$$
$$= \boxed{-15} \div \boxed{+11} \frac{+1}{\underbrace{x \cdot x \cdot x \cdot x}_{9 - 5 \text{ copies of } x}}$$

iv. We recode

$$= \boxed{-15} \div \boxed{+11}\, x^{-\, 9-5}$$
$$= \boxed{-1.36+[...]}\, x^{-\, 4}$$

In the following examples, we will have to use Theorem 11.2 Reciprocal because we will be $\boxdiv$'ing by a Laurent monomial with a negative exponent, that is a Laurent monomial in which the coefficient is divided by the copies of x.

EXAMPLE 11.14. To execute $\left[-15\, x^{+5}\right] \boxed{\div} \left[+11\, x^{-2}\right]$

i. StartIng from the specifying code

$$\left[-15\, x^{+5}\right] \boxed{\div} \left[+11\, x^{-2}\right] =$$

ii. We decode:

$$= \left[-15 \text{ multiplied by } 5 \text{ copies of } x\right] \div \left[+11 \text{ divided by } 2 \text{ copies of } x\right]$$

iii. We execute

$$= \left[-15 \cdot \underbrace{x \cdot x \cdot x \cdot x \cdot x}_{5 \text{ copies of } x}\right] \div \left[\frac{+11}{\underbrace{x \cdot x}_{2 \text{ copies of } x}}\right]$$

Here we use Theorem 11.2 Reciprocal on page 7

$$= \left[-15 \cdot \underbrace{x \cdot x \cdot x \cdot x \cdot x}_{5 \text{ copies of } x}\right] \cdot \left[\frac{\underbrace{x \cdot x}_{2 \text{ copies of } x}}{+11}\right]$$

$$= -15 \div +11 \cdot \underbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}_{5+2 \text{ copies of } x}$$

iv. We recode

$$= -15 \div +11\, x^{+\,5+2}$$

$$= -1.36+[...]\, x^{+7}$$

EXAMPLE 11.15. To execute $\left[-15 \cdot x^{-3}\right] \boxed{\div} \left[+11 \cdot x^{-7}\right]$

i. StartIng from the specifying code

$$\left[-15\, x^{-3}\right] \boxed{\div} \left[+11\, x^{-7}\right] =$$

ii. We decode:

$$= \left[-15 \text{ divided by } 3 \text{ copies of } x\right] \div \left[+11 \text{ divided by } 7 \text{ copies of } x\right]$$

iii. We execute

$$= \left[\frac{\boxed{-15}}{\underbrace{x \cdot x \cdot x}_{\boxed{3}\text{ copies of } x}} \right] \div \left[\frac{\boxed{+11}}{\underbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}_{\boxed{7}\text{ copies of } x}} \right]$$

Here we use Theorem 11.2 Reciprocal on page 7:

$$= \left[\frac{\boxed{-15}}{\underbrace{x \cdot x \cdot x}_{\boxed{3}\text{ copies of } x}} \right] \cdot \left[\frac{\underbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}_{\boxed{7}\text{ copies of } x}}{\boxed{+11}} \right]$$

$$= \boxed{-15} \div \boxed{+11} \frac{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x}$$

$$= \boxed{-15} \div \boxed{+11} \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}}$$

$$= \boxed{-15} \div \boxed{+11} \frac{x \cdot x \cdot x \cdot x}{+1}$$

$$= \boxed{-15} \div \boxed{+11} \boxed{\cdot} \underbrace{x \cdot x \cdot x \cdot x}_{\boxed{7} - \boxed{3}\text{ copies of } x}$$

iv. We recode

$$= \boxed{-15} \div \boxed{+11}\, x^{\boxed{+}\boxed{7-3}}$$

$$= \boxed{-1.36+[...]}\, x^{\boxed{+}\boxed{4}}$$

4. We saw in the above examples that *dividing* Laurent monomials using Definition 11.1 on page 1 is long and painful. But since we used $\boxed{+}$ to code for $\boxed{\text{multiplied by}}$ and $\boxed{-}$ to code for $\boxed{\text{divided by}}$ so that the exponents are signed counting numerators, it so happens that $\ominus$ automatically takes care of everything in all cases:

> **THEOREM 11.3 Laurent Monomial Division**
> In order to *divide* two monomials $ax^{\pm m}$ and $bx^{\pm n}$,
> - We *divide* the coefficients,
> - We *ominus* the exponents.
>
> In other words:
> $$ax^{\pm m} \boxdiv bx^{\pm n} = a \div b\, x^{\pm m \ominus \pm n}$$

EXAMPLE 11.16. To execute $\left[-17 \cdot x^{+7} \right] \boxdiv \left[+11 \cdot x^{+4} \right]$ using Theorem 11.3 Laurent Monomial Division on page 10 we just write

$$
\begin{aligned}
\left[-17 \cdot x^{+7} \right] \boxdiv \left[+11 \cdot x^{+4} \right] &= \\
&= -17 \div +11 \, x^{+7 \ominus +4} \\
&= -17 \div +11 \, x^{+7 \oplus -4} \\
&= -1.54+[...] \, x^{+3}
\end{aligned}
$$

which is of course the same result at in Example 11.12 on page 7.

EXAMPLE 11.17. To execute $\left[-15 \cdot x^{+5} \right] \boxdiv \left[+11 \cdot x^{+9} \right]$ using Theorem 11.3 Laurent Monomial Division on page 10 we just write:

$$
\begin{aligned}
\left[-15 \cdot x^{+5} \right] \boxdiv \left[+11 \cdot x^{+9} \right] &= \\
&= -15 \div +11 \, x^{+5 \ominus +9} \\
&= -15 \div +11 \, x^{+5 \oplus -9} \\
&= -1.36+[...] \, x^{-4}
\end{aligned}
$$

which is of course the same result at in Example 11.13 on page 8.

EXAMPLE 11.18. To execute $\left[-15 \cdot x^{+5} \right] \boxdiv \left[+11 \cdot x^{-2} \right]$ using Theorem 11.3 Laurent Monomial Division on page 10 we just write:

$$
\begin{aligned}
\left[-15 \cdot x^{+5} \right] \boxdiv \left[+11 \cdot x^{-2} \right] &= \\
&= -15 \div +11 \, x^{+5 \ominus -2} \\
&= -15 \div +11 \, x^{+5 \oplus +2} \\
&= -1.36+[...] \, x^{+7}
\end{aligned}
$$

which is of course the same result at in Example 11.14 on page 9.

EXAMPLE 11.19. To execute $\left[-15 \cdot x^{-3} \right] \boxdiv \left[+11 \cdot x^{-7} \right]$ using Theorem 11.3 Laurent Monomial Division on page 10 we just write

$$
\left[-15 \cdot x^{-3} \right] \boxdiv \left[+11 \cdot x^{-7} \right] =
$$

$$= \boxed{-15} \div \boxed{+11}\, x^{\boxed{-}\boxed{3} \ominus \boxed{-}\boxed{7}}$$
$$= \boxed{-15} \div \boxed{+11}\, x^{\boxed{-}\boxed{3} \oplus \boxed{+}\boxed{7}}$$
$$= \boxed{-1.36+[...]}\, x^{\boxed{+}\boxed{4}}$$

which is of course the same result at in Example 11.15 on page 9.

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