Chapter 13

Multiplication of Polynomials

Multiplication of polynomials is very close to multiplication of decimal number-phrases so, we begin by discussion of multiplication in ARITHMETIC.

13.1 Multiplication in Arithmetic

In ARITHMETIC, multiplication is an operation that is very different from addition in many ways.

1. While number-phrases (with a common denominator) can always be added, number-phrases, even with a common denominator, usually cannot be multiplied.

   Example 13.1.
   While
   
   \[2 \text{ Apples} + 3 \text{ Apples} = 5 \text{ Apples}\]
   
   the following
   
   \[2 \text{ Apples} \times 3 \text{ Apples}\]
   
   makes no sense whatsoever.
   (\(2 \text{ Apples} \times 3 \text{ Apples}\) is not the same as \(2(3 \text{ Apples})\) which is equal to \(6 \text{ Apples}\))

2. Even when number-phrases can be multiplied, the result involves a different denominator.

   Example 13.2.
   • Say that, in the real world, we want to tile a table three feet long by two feet wide
Chapter 13. Multiplication of Polynomials

We need three tiles to tile the first row:

and another three tiles to tile the second row:

Altogether then, we used six one foot by one foot tiles.

- On paper, the specifying-phrase that represents the area of the table is
  \(2\ \text{Feet} \times 3\ \text{Feet}\)
  and the number-phrase that represents the area of a tile is
  \(1\ \text{FootByFoot}\)
  also known as
  \(1\ \text{SquareFoot}\)
  We then represent the fact that we used two rows of three tiles by

\[
2(3\ \text{FootByFoot}) = (2 \times 3)\ \text{FootByFoot} = 6\ \text{FootByFoot}
\]

Altogether, we represent the real world tiling process by the paper procedure

\[
2\ \text{Feet} \times 3\ \text{Feet} = (2 \times 3)\ \text{FootByFoot}
\]

3. While number-phrases involving different denominators can never be added, number-phrases involving different denominators can occasionally be multiplied.

**Example 13.3.**
13.2 Multiplication of Polynomials

- Say that, in the real world, we want to tile a shelf three feet long by six inches wide with one foot by one inch tiles.

We need three tiles to tile each row and since there are six rows we need eighteen one foot by one inch tiles.

- On paper, the specifying-phrase that represents the area of the table is 3 Feet × 6 Inches and the number-phrase that represents the area of a tile is 1 FootByInch.

We then represent the fact that we used six rows of three tiles by

\[ 6(3 \text{ FootByInch}) = (6 \times 3) \text{ FootByInch} = 18 \text{ FootByInch} \]

Altogether, we represent the real world tiling process by the paper procedure

\[ 6 \text{ Feet} \times 3 \text{ Inches} = (6 \times 3) \text{ FootByInch} \]

13.2 Multiplication of Polynomials

In polynomial algebra, things are much simpler: Because we can always multiply monomials, it turns out that we can multiply polynomials. We will use the symbol \( \times \) to denote multiplication of polynomials.

1. In order to multiply a given polynomial by a given monomial, we multiply each and every monomial in the given polynomial by the given monomial and the result is another polynomial.

Example 13.4. Given the polynomial

\[ +2x^2 + 4x + 6x^0 - 6x^{-1} - 5x^{-2} \]
Chapter 13. Multiplication of Polynomials

and the monomial \(-4x^3\)

In order to identify the specifying phrase

\[
\left[ +2x^2 + 4x^1 + 6x^0 - 6x^{-1} - 5x^{-2} \right] \times \left[ -4x^3 \right]
\]

i. We set up as in arithmetic

\[
+2x^2 + 4x^1 + 6x^0 - 6x^{-1} - 5x^{-2} \\
-9x^2
\]

\[
\begin{array}{r}
  +2x^2 + 4x^1 + 6x^0 - 6x^{-1} - 5x^{-2} \\
  -9x^2
  \end{array}
\]

\[
(+2)(-9)x^{2+3} + (4)(-9)x^{1+3} + (6)(-9)x^{0+3} + (-6)(-9)x^{-1+3} + (-5)(-9)x^{-2+3}
\]

ii. We multiply each and every monomial in the given polynomial by the given monomial:

\[
+2x^2 + 4x^1 + 6x^0 - 6x^{-1} - 5x^{-2} \\
-9x^2
\]

\[
\begin{array}{r}
  +2x^2 + 4x^1 + 6x^0 - 6x^{-1} - 5x^{-2} \\
  -9x^2
  \end{array}
\]

\[
-18x^4 - 36x^3 - 54x^2 - 54x^1 + 45x^0
\]

2. In order to multiply a first polynomial by a second polynomial, we multiply each and every monomial in the first polynomial by each and every monomial in the first polynomial and the result is another polynomial. In order to keep some order in the procedure,

i. We set up the multiplication pretty much as in arithmetic:

a. We write the first polynomial on the first line with missing monomials written-in with a 0 coefficient

b. We write the second polynomial on the second line without writing-in the missing monomials with a 0 coefficient. Also, the second polynomial need not be lined up exponent-wise with the first polynomial

ii. We write the results of the multiplication of the first polynomial by each monomial of the second polynomial on a separate line

iii. As we write the results of the multiplication of the first polynomial by the next monomial of the second polynomial on a separate line

iv. We add the terms with same exponent (lined up vertically as a result of the previous step).

**Example 13.5.** Given a first polynomial

\[
+5x^3 - 4x^1 + 6x^0 - 7x^{-2}
\]
and a second polynomial
\[ +2x^2 - 8x^1 + 3x^{-1} \]

In order to identify the specifying phrase
\[ +5x^3 - 4x^1 + 6x^0 - 7x^{-2} \]
we proceed as follows:

\[ \text{i. We set up as usual, writing the monomials missing in the first polynomial with a 0 coefficient.} \]
\[ +5x^3 + 0x^2 - 4x^1 + 6x^0 + 0x^{-1} - 7x^{-2} \]
\[ + 2x^2 - 8x^1 + 3x^{-1} \]

\[ \text{ii. We multiply each and every monomial in the first polynomial by the first monomial in the second polynomial, writing the missing monomials with a 0 coefficient.} \]
\[ +5x^3 + 0x^2 - 4x^1 + 6x^0 + 0x^{-1} - 7x^{-2} \]
\[ + 2x^2 - 8x^1 + 3x^{-1} \]
\[ +10x^5 + 0x^4 - 8x^3 + 12x^2 + 0x^1 - 14x^0 \]

\[ \text{iii. We multiply each and every monomial in the first polynomial by the second monomial in the second polynomial, writing the missing monomials with a 0 coefficient.} \]
\[ +5x^3 + 0x^2 - 4x^1 + 6x^0 + 0x^{-1} - 7x^{-2} \]
\[ + 2x^2 - 8x^1 + 3x^{-1} \]
\[ +10x^5 + 0x^4 - 8x^3 + 12x^2 + 0x^1 - 14x^0 \]
\[ -40x^4 + 0x^3 + 32x^2 - 48x^1 + 0x^0 + 56x^{-1} \]

\[ \text{iv. We multiply each and every monomial in the first polynomial by the third monomial in the second polynomial, writing the missing monomials with a 0 coefficient.} \]
\[ +5x^3 + 0x^2 - 4x^1 + 6x^0 + 0x^{-1} - 7x^{-2} \]
\[ + 2x^2 - 8x^1 + 3x^{-1} \]
\[ +10x^5 + 0x^4 - 8x^3 + 12x^2 + 0x^1 - 14x^0 \]
\[ -40x^4 + 0x^3 + 32x^2 - 48x^1 + 0x^0 + 56x^{-1} \]
\[ +15x^2 + 0x^1 - 12x^0 + 18x^{-1} + 0x^{-2} - 21x^{-3} \]

\[ \text{v. We add the terms with same exponent} \]
\[ +5x^3 + 0x^2 - 4x^1 + 6x^0 + 0x^{-1} - 7x^{-2} \]
\[ + 2x^2 - 8x^1 + 3x^{-1} \]
\[ +10x^5 + 0x^4 - 8x^3 + 12x^2 + 0x^1 - 14x^0 \]
\[ -40x^4 + 0x^3 + 32x^2 - 48x^1 + 0x^0 + 56x^{-1} \]
\[ +15x^2 + 0x^1 - 12x^0 + 18x^{-1} + 0x^{-2} - 21x^{-3} \]
<table>
<thead>
<tr>
<th>$+15x^2$</th>
<th>$+0x^1$</th>
<th>$-12x^0$</th>
<th>$+18x^{-1}$</th>
<th>$+0x^{-2}$</th>
<th>$-21x^{-3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+10x^5$</td>
<td>$-40x^4$</td>
<td>$-8x^3$</td>
<td>$+59x^2$</td>
<td>$-48x^1$</td>
<td>$-26x^0$</td>
</tr>
</tbody>
</table>

**v.** We thus have:

\[
\left[ +5x^3 - 4x^1 + 6x^0 - 7x^{-2} \right] \times \left[ +2x^2 - 8x^1 + 3x^{-1} \right] = \\
= +10x^5 - 40x^4 - 8x^3 + 59x^2 - 48x^1 - 26x^0 + 74x^{-1} + 0x^{-2} - 21x^{-3}
\]