Chapter 13

Multiplication of Polynomials

Multiplication of polynomials is very close to multiplication of decimal number-phrases so, we begin by discussion of multiplication in ARITHMETIC.

13.1 Multiplication in Arithmetic

In ARITHMETIC, *multiplication* is an operation that is very different from *addition* in many ways.

1. While number-phrases (with a common denominator) can always be added, number-phrases, even with a common denominator, usually cannot be multiplied.

EXAMPLE 13.1.
While

$$2 \textsf{ Apples} + 3 \textsf{ Apples} = 5 \textsf{ Apples}$$

the following

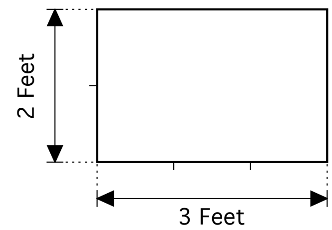
$$2 \textsf{ Apples} \times 3 \textsf{ Apples}$$

makes no sense whatsoever.
(2 **Apples** $\times$ 3 **Apples** is not the same as $2(3$ **Apples**$)$ which is equal to 6 **Apples**)

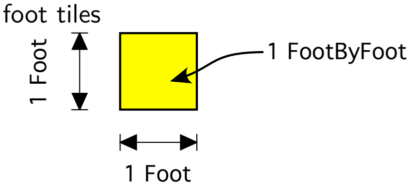
2. Even when number-phrases can be multiplied, the result involves a *different* denominator.

EXAMPLE 13.2.

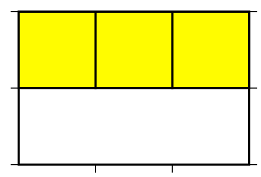
- Say that, in the real world, we want to tile a table three feet long by two feet wide



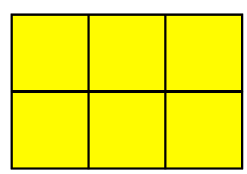
with one foot by one foot tiles



We need three tiles to tile the first row:



and another three tiles to tile the second row:



Altogether then, we used six one foot by one foot tiles.

- On paper, the *specifying-phrase* that represents the area of the *table* is

$$2 \textbf{ Feet} \times 3 \textbf{ Feet}$$

and the *number-phrase* that represents the area of a *tile* is

$$1 \textbf{ FootByFoot}$$

also known as

$$1 \textbf{ SquareFoot}$$

We then represent the fact that we used two rows of three tiles by

$$\begin{aligned} 2(3 \textbf{ FootByFoot}) &= (2 \times 3) \textbf{ FootByFoot} \\ &= 6 \textbf{ FootByFoot} \end{aligned}$$

Altogether, we represent the real world tiling *process* by the paper *procedure*

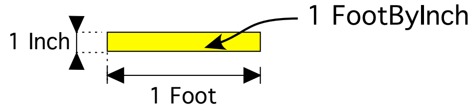
$$2 \textbf{ Feet} \times 3 \textbf{ Feet} = (2 \times 3) \textbf{ FootByFoot}$$

3. While number-phrases involving different denominators can never be added, number-phrases involving different denominators can occasionally be multiplied.

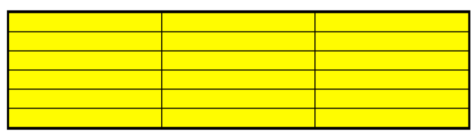
EXAMPLE 13.3.

- Say that, in the real world, we want to tile a shelf three feet long by six inches wide

with one foot by one inch tiles



We need three tiles to tile each row and since there are six rows



we need eighteen one foot by one inch tiles.

- On paper, the *specifying-phrase* that represent the area of the *table* is

$$3\ \textbf{Feet} \times 6\ \textbf{Inches}$$

and the *number-phrase* that represents the area of a *tile* is

$$1\ \textbf{FootByInch}$$

We then represent the fact that we used six rows of three tiles by

$$\begin{aligned} 6(3\ \textbf{FootByInch}) &= (6 \times 3)\ \textbf{FootByInch} \\ &= 18\ \textbf{FootByInch} \end{aligned}$$

Altogether, we represent the real world tiling *process* by the paper *procedure*

$$6\ \textbf{Feet} \times 3\ \textbf{Inches} = (6 \times 3)\ \textbf{FootByInch}$$

13.2 Multiplication of Polynomials

In POLYNOMIAL ALGEBRA, things are much simpler: Because we can always multiply monomials, it turns out that we can multiply polynomials. We will use the symbol $\boxtimes$ to denote multiplication of polynomials.

1. In order to multiply a given polynomial by a given monomial, we multiply each and every monomial in the given polynomial by the given monomial and the result is another polynomial.

EXAMPLE 13.4. Given the polynomial

$$+2x^{+2} + 4x^{+1} + 6x^{0} - 6x^{-1} - 5x^{-2}$$

and the monomial

$$-4x^{+3}$$

In order to identify the specifying phrase

$$\left[+2x^{+2}+4x^{+1}+6x^{0}-6x^{-1}-5x^{-2}\right]\boxtimes\left[-4x^{+3}\right]$$

i. We set up as in ARITHMETIC

$$\begin{array}{rl} & +2x^{+2}+4x^{+1}+6x^{0}-6x^{-1}-5x^{-2} \\ \boxtimes & \qquad\qquad -9x^{+2} \\ \hline \end{array}$$

ii. We multiply each and every monomial in the given polynomial by the given monomial:

$$\begin{array}{rl} & +2x^{+2}+4x^{+1}+6x^{0}-6x^{-1}-5x^{-2} \\ \boxtimes & \qquad\qquad -9x^{+2} \\ \hline \end{array}$$

$$(+2)(-9)x^{+2\oplus+2}(+4)(-9)x^{+1\oplus+2}(+6)(-9)x^{0\oplus+2}(-6)(-9)x^{-1\oplus+2}(-5)(-9)x^{-2\oplus+2}$$

iii. We get

$$\begin{array}{rl} & +2x^{+2}+4x^{+1}+6x^{0}-6x^{-1}-5x^{-2} \\ \boxtimes & \qquad\qquad -9x^{+2} \\ \hline \end{array}$$

$$-18x^{+4}-36x^{+3}-54x^{+2}-54x^{+1}+45x^{0}$$

2. In order to multiply a first polynomial by a second polynomial, we multiply each and every monomial in the first polynomial by each and every monomial in the first polynomial and the result is another polynomial.
In order to keep some order in the procedure,
i. We set up the multiplication pretty much as in ARITHMETIC:

a. We write the first polynomial on the first line with missing monomials written-in with a 0 coefficient

b. We write the second polynomial on the second line *without* writing-in the missing monomials with a 0 coefficient. Also, the second polynomial need not be lined up exponent-wise with the first polynomial

ii. We write the results of the multiplication of the first polynomial by each monomial of the second polynomial on a separate line
iii. As we write the results of the multiplication of the first polynomial by the next monomial of the second polynomial, we make sure that the exponents are lined up vertically (to make the next step easier).
iv. We add the terms with same exponent (lined up vertically as a result of the previous step).

EXAMPLE 13.5. Given a first polynomial

$$+5x^{+3}-4x^{+1}+6x^{0}-7x^{-2}$$

and a second polynomial

$$+2x^{+2} - 8x^{+1} + 3x^{-1}$$

In order to identify the specifying phrase

$$\left[+5x^{+3} \ - 4x^{+1} \ + 6x^{0} \ - 7x^{-2}\right] \boxtimes \left[+2x^{+2} \ - 8x^{+1} \ + 3x^{-1}\right]$$

we proceed as follows:

i. We set up as usual, writing the monomials missing *in the first polynomial* with a 0 coefficient.

$$\begin{array}{l} \quad +5x^{+3} + 0x^{+2} - 4x^{+1} + 6x^{0} + 0x^{-1} - 7x^{-2} \\ \boxtimes \qquad\qquad + 2x^{+2} - 8x^{+1} + 3x^{-1} \\ \hline \end{array}$$

ii. We multiply each and every monomial in the first polynomial by the first monomial in the second polynomial, writing the missing monomials with a 0 coefficient.

$$\begin{array}{l} \quad +5x^{+3} + 0x^{+2} - 4x^{+1} + 6x^{0} + 0x^{-1} - 7x^{-2} \\ \boxtimes \qquad\qquad +2x^{+2} - 8x^{+1} + 3x^{-1} \\ \hline +10x^{+5} + 0x^{+4} - 8x^{+3} + 12x^{+2} + 0x^{+1} - 14x^{0} \end{array}$$

iii. We multiply each and every monomial in the first polynomial by the second monomial in the second polynomial, writing the missing monomials with a 0 coefficient.

$$\begin{array}{l} \quad +5x^{+3} + 0x^{+2} - 4x^{+1} + 6x^{0} + 0x^{-1} - 7x^{-2} \\ \boxtimes \qquad\qquad + 2x^{+2} - 8x^{+1} + 3x^{-1} \\ \hline +10x^{+5} + 0x^{+4} - 8x^{+3} + 12x^{+2} + 0x^{+1} - 14x^{0} \\ \qquad\qquad -40x^{+4} + 0x^{+3} + 32x^{+2} \ - 48x^{+1} + 0x^{0} \ + 56x^{-1} \end{array}$$

iv. We multiply each and every monomial in the first polynomial by the third monomial in the second polynomial, writing the missing monomials with a 0 coefficient.

$$\begin{array}{l} \quad +5x^{+3} + 0x^{+2} - 4x^{+1} + 6x^{0} + 0x^{-1} - 7x^{-2} \\ \boxtimes \qquad\qquad + 2x^{+2} - 8x^{+1} + 3x^{-1} \\ \hline +10x^{+5} + 0x^{+4} - 8x^{+3} + 12x^{+2} + 0x^{+1} - 14x^{0} \\ \qquad\qquad -40x^{+4} + 0x^{+3} + 32x^{+2} \ - 48x^{+1} + 0x^{0} \ + 56x^{-1} \\ \qquad\qquad\qquad\qquad\qquad\qquad +15x^{+2} \ + 0x^{+1} \ - 12x^{0} \ + 18x^{-1} \ + 0x^{-2} \ - 21x^{-3} \end{array}$$

v. We add the terms with same exponent

$$\begin{array}{l} \quad +5x^{+3} + 0x^{+2} - 4x^{+1} + 6x^{0} + 0x^{-1} - 7x^{-2} \\ \boxtimes \qquad\qquad + 2x^{+2} - 8x^{+1} + 3x^{-1} \\ \hline +10x^{+5} + 0x^{+4} - 8x^{+3} + 12x^{+2} + 0x^{+1} - 14x^{0} \\ \qquad\qquad -40x^{+4} + 0x^{+3} + 32x^{+2} \ - 48x^{+1} + 0x^{0} \ + 56x^{-1} \end{array}$$

$$\begin{array}{rrrrrrrrr} & & & +15x^{+2} & +0x^{+1} & -12x^{0} & +18x^{-1} & +0x^{-2} & -21x^{-3} \\ \hline +10x^{+5} & -40x^{+4} & -8x^{+3} & +59x^{+2} & -48x^{+1} & -26x^{0} & +74x^{-1} & +0x^{-2} & -21x^{-3} \end{array}$$

v. We thus have:

$$\left[+5x^{+3} - 4x^{+1} + 6x^{0} - 7x^{-2}\right] \boxtimes \left[+2x^{+2} - 8x^{+1} + 3x^{-1}\right] =$$
$$= +10x^{+5} - 40x^{+4} - 8x^{+3} + 59x^{+2} - 48x^{+1} - 26x^{0} + 74x^{-1} + 0x^{-2} - 21x^{-3}$$