Chapter 14

Division of Polynomials

We now turn to the last one of the four operation with polynomials: division. However, in order to understand the procedure, we must first take a look at the division procedure in ARITHMETIC.

14.1 Division In Descending Powers

Since *decimal numerators* are combinations of powers of TEN, it should not be surprising that the above procedure should work for *polynomials* which are combinations of powers of x.

The *procedure* consists of successive *cycles*, one for each monomial in the quotient. During each of these *cycles*, we go through four *step*s:

Step I. We find each *monomial* of the *quotient* by dividing the *first monomial* in the divisor into the *first monomial* of the previous partial remainder.

Step II. We find the *partial product* by multiplying the *full divisor* by the *monomial* of the quotient we found in Step I.

Step III. We find the *partial remainder* by subtracting the *partial product* we found in Step II from the previous partial remainder or, if there is not yet a partial remainder, from the *full dividend*.

Step IV. We decide if we

- *stop* the division
- *continue* the division

Just as, in ARITHMETIC, we can stop the division anywhere we want and we need not stop a division when the quotient reaches a monomial with exponent 0 because we can always divide a monomial into another since we can have *negative* exponents. In fact, again just as in ARITHMETIC, there are cases where we absolutely need to go beyond the exponent 0 and use negative exponents. (See Epilogue.)

EXAMPLE 14.1. In order to compute $\dfrac{-12x^3+11x^2-17x+1}{-3x^2+5-2}$, we divide $-3x^2+5-2$ into $-12x^3+11x^2-17x+1$:

$$\begin{array}{rrr|rrrr} -3x^2 & +5x & -2 & -12x^3 & +11x^2 & -17x & +1 \end{array}$$

we proceed as follows:

CYCLE 1. Step I. We find the *first monomial in the quotient* by dividing the *first monomial in the divisor*, $-3x^2$, into the *first monomial of the dividend*, $-12x^3$ that is $\dfrac{-12x^3}{-3x^2} = +4x$

$$\begin{array}{rrr|rrrr} & & & +4x & & & \\ \hline -3x^2 & +5x & -2 & -12x^3 & +11x^2 & -17x & +1 \end{array}$$

Step II. We find the *first partial product* by multiplying the *full divisor* by the *first monomial in the quotient*:

$$\begin{array}{lrrr|rrrr} & & & & +4x & & & \\ \hline & -3x^2 & +5x & -2 & -12x^3 & +11x^2 & -17x & +1 \\ \text{First partial product:} & & & & -12x^3 & +20x^2 & -8x & \end{array}$$

Step III. We find the *first partial remainder* by *subtracting* the first partial product from the full dividend:

$$\begin{array}{rrr|rrrrr} & & & & +4x & & & \\ \hline -3x^2 & +5x & -2 & & -12x^3 & +11x^2 & -17x & +1 \\ & & & \ominus & -12x^3 & +20x^2 & -8x & \end{array}$$

But to *subtract* the first partial product means to *add the opposite* of the first partial product to the full dividend:

$$\begin{array}{lrrr|rrrrr} & & & & & +4x & & & \\ \hline & -3x^2 & +5x & -2 & & -12x^3 & +11x^2 & -17x & +1 \\ & & & & \oplus & +12x^3 & -20x^2 & +8x & \\ \hline \text{First remainder:} & & & & & +0x^3 & -9x^2 & -9x & +1 \end{array}$$

Step IV. We decide if we want to *stop* or *continue* the division.

- If we decide to *stop* the division,
  - the *quotient* of the division is $+4x$.
  - the *remainder* of the division is $-9x^2 - 8x + 1$

  If we don't care about the *remainder*, we write:

$$\frac{-12x^3+11x^2-17x+1}{-3x^2+5x-2} = +4x+(\ldots)$$

where we write $+\ (\ldots)$ as a reminder that $\dfrac{-12x^3+11x^2-17x+1}{-3x^2+5x-2}$ is not exactly equal to $+4x$ since there was a *remainder*.

- If we decide to *continue* the division, we start a new cycle

CYCLE 2. Step I. We find the *second monomial in the quotient* by dividing the *first monomial in the divisor*, $-3x^2$, into *the first monomial in the first partial remainder*, $-9x^2$, that is $\dfrac{-9x^2}{-3x^2} = +3$

$$\begin{array}{lll|llll} & & & +4x & +3 & & \\ \cline{4-7} -3x^2 & +5x & -2\,) & -12x^3 & +11x^2 & -17x & +1 \\ & & & -12x^3 & +20x^2 & -8x & \\ \cline{4-7} & & & & -9x^2 & -9x & +1 \end{array}$$

Step II. We find the *second partial product* by multiplying the *full divisor* by the *second monomial in the quotient*:

$$\begin{array}{lll|llll} & & & +4x & +3 & & \\ \cline{4-7} -3x^2 & +5x & -2\,) & -12x^3 & +11x^2 & -17x & +1 \\ & & & -12x^3 & +20x^2 & -8x & \\ \cline{4-7} & & & & -9x^2 & -9x & +1 \\ \text{Second partial product:} & & & & -9x^2 & +15x & -6 \end{array}$$

Step III. We find the *second partial remainder* by *subtracting* the second partial product from the first partial remainder:

$$\begin{array}{lll|llll} & & & +4x & +3 & & \\ \cline{4-7} -3x^2 & +5x & -2\,) & -12x^3 & +11x^2 & -17x & +1 \\ & & & -12x^3 & +20x^2 & -8x & \\ \cline{4-7} & & & & -9x^2 & -9x & +1 \\ & & & \ominus & -9x^2 & +15x & -6 \\ \cline{5-7} \end{array}$$

But to *subtract* the second partial product means to *add the opposite* of the second partial product to the first partial remainder:

$$\begin{array}{lll|llll} & & & +4x & +3 & & \\ \cline{4-7} -3x^2 & +5x & -2\,) & -12x^3 & +11x^2 & -17x & +1 \\ & & & -12x^3 & +20x^2 & -8x & \\ \cline{4-7} & & & & -9x^2 & -9x & +1 \\ & & & \oplus & +9x^2 & -15x & +6 \\ \cline{5-7} \text{Second remainder:} & & & & +0x^2 & -24x & +7 \end{array}$$

Step IV. We decide if we want to *stop* or *continue* the division.

- If we decide to *stop* the division,
  - the *quotient* of the division is $+4x+3$.
  - the *remainder* of the division is $-24x+7$

  If we don't care about the *remainder*, we write:

$$\frac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5x - 2} = +4x + 3 + (\dots)$$

where we write $+ (\dots)$ as a reminder that $\dfrac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5x - 2}$ is not exactly equal to $+4x + 3$ since there was a *remainder*.

- If we decide to *continue* the division, we start a new cycle

CYCLE 3. Step I. We find the *third monomial in the quotient* by dividing the *first monomial in the divisor*, $-3x^2$, into *the first monomial in the second partial remainder*, $-24x$ that is $\dfrac{-24x}{-3x^2} = +8x^{-1}$

$$\begin{array}{rr|rrrrr}
 & & +4x & +3 & +8x^{-1} & & \\ \hline
-3x^2 & +5x\ -2 & -12x^3 & +11x^2 & -17x & +1 & \\
 & & -12x^3 & +20x^2 & -8x & & \\ \hline
 & & & -9x^2 & -9x & +1 & \\
 & & & +9x^2 & -15x & +6 & \\ \hline
 & & & & -24x & +7 &
\end{array}$$

Step II. We find the *third partial product* by multiplying the *full divisor* by the *third monomial in the quotient*:

$$\begin{array}{rr|rrrrr}
 & & +4x & +3 & +8x^{-1} & & \\ \hline
-3x^2 & +5x\ -2 & -12x^3 & +11x^2 & -17x & +1 & \\
 & & -12x^3 & +20x^2 & -8x & & \\ \hline
 & & & -9x^2 & -9x & +1 & \\
 & & & +9x^2 & -15x & +6 & \\ \hline
 & & & & -24x & +7 & \\
\text{Third partial product:} & & & & -24x & +40 & -16x^{-1}
\end{array}$$

Step III. We find the *third partial remainder* by *subtracting* the third partial product from the first partial remainder:

$$\begin{array}{rr|rrrrr}
 & & +4x & +3 & +8x^{-1} & & \\ \hline
-3x^2 & +5x\ -2 & -12x^3 & +11x^2 & -17x & +1 & \\
 & & -12x^3 & +20x^2 & -8x & & \\ \hline
 & & & -9x^2 & -9x & +1 & \\
 & & & +9x^2 & -15x & +6 & \\ \hline
 & & & & -24x & +7 & \\
 & & \ominus & & -24x & +40 & -16x^{-1}
\end{array}$$

But to *subtract* the second partial product means to *add the opposite* of the second partial product to the first partial remainder:

$$
\begin{array}{rrrrrr}
 & +4x & +3 & +8x^{-1} & & \\
\hline
-3x^2 +5x -2 \;) & -12x^3 & +11x^2 & -17x & +1 & \\
 & -12x^3 & +20x^2 & -8x & & \\
\hline
 & & -9x^2 & -9x & +1 & \\
 & & +9x^2 & -15x & +6 & \\
\hline
 & & & -24x & +7 & \\
 & \oplus & & +24x & -40 & +16x^{-1} \\
\hline
\text{Third remainder:} & & & 0x & -33 & +16x^{-1}
\end{array}
$$

Step IV. We decide if we want to *stop* or *continue* the division.

- If we decide to *stop* the division,
  - the *quotient* of the division is $+4x + 3 + 8x^{-1}$.
  - the *remainder* of the division is $-33 + 16x^{-1}$

  If we don't care about the *remainder*, we write:

  $$\frac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5x - 2} = +4x + 3 + 8x^{-1} + (\ldots)$$

  where we write $+ (\ldots)$ as a reminder that $\dfrac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5x - 2}$ is not exactly equal to $+4x + 3 + 8x^{-1}$ since there was a *remainder*.
- If we decide to *continue* the division, we start a new cycle

The procedure to divide polynomials is in fact a lot simpler than the procedure for dividing in ARITHMETIC:

- There is never any "carryover"
- The first term of each partial remainder has coefficient 0
- There are no Trials in **Step I** because, when we divide the first monomial in the divisor into the first monomial of a partial remainder, we always get a coefficient for the corresponding monomial in the quotient and the worst that can happen is that this coefficient is a fraction.

EXAMPLE 14.2. In order to divide $2x^3 + 5x^2 - 6$ by $3x - 1$ we write (in the *anglo-saxon* tradition):

$$
\begin{array}{rrrrr}
 & & \frac{2}{3}x^2 & + \frac{17}{9}x & + \frac{17}{27} \\
\hline
3x-1\,) & 2x^3 & +5x^2 & & -6 \\
 & -2x^3 & + \frac{2}{3}x^2 & & \\
\hline
 & & \frac{17}{3}x^2 & & \\
 & & -\frac{17}{3}x^2 & + \frac{17}{9}x & \\
\hline
 & & & \frac{17}{9}x & -6 \\
 & & & -\frac{17}{9}x & + \frac{17}{27} \\
\hline
 & & & & -\frac{145}{27}
\end{array}
$$

The *quotient* is

$$+\tfrac{2}{3}x^2 + \tfrac{17}{9}x + \tfrac{17}{27}$$

The *remainder* is

$$-\tfrac{145}{27}$$

14.2 Default Rules for Division

Since mathematicians are lazy, mathematicians want to write only one stage in Step III but there are two traditions concerning what then to write, as a result, in Step II:

- In the *latin* tradition, in Step II, we write the *partial product*, that is what we get it from the *multiplication*, and so in Step III, when it comes to subtracting, we visualize the *opposite of the partial product* we wrote in Step II and we oplus what we *visualize*. The advantage is that each line is exactly what we get from the previous operation.

  EXAMPLE 14.3.

$$\begin{array}{lll|rrrr} & & & +4x & +3 & & \\ \hline -3x^2 & +5x & -2 & -12x^3 & +11x^2 & -16x & +1 \\ & & & -12x^3 & +20x^2 & -8x & \\ \cline{4-7} & & & & -9x^2 & -8x & +1 \end{array}$$

- In the *anglo-saxon* tradition, we anticipate the subtraction to be done in Step III and in Step II *we write the opposite of the partial product* so in Step III we oplus what we *wrote* in Step II.

  EXAMPLE 14.4.

$$\begin{array}{lll|rrrr} & & & +4x & +3 & & \\ \hline -3x^2 & +5x & -2 & -12x^3 & +11x^2 & -16x & +1 \\ & & & +12x^3 & -20x^2 & +8x & \\ \cline{4-7} & & & & -9x^2 & -8x & +1 \end{array}$$

  From now on we will of course follow the *anglo-saxon* tradition.

EXAMPLE 14.5. In order to compute $\dfrac{6x^3 + 13x^2 + 13x + 7}{2x + 1}$, we divide $2x + 1$ into $6x^3 + 13x^2 + 13x + 7$:

CYCLE 1. Step I. We find the *first monomial* in the quotient by *short division*:

$$\begin{array}{r|l} & 3x^2 \\ \hline 2x+1 & 6x^3 + 13x^2 + 13x + 7 \end{array}$$

Step II. We get the *first opposite product* by writing the opposite of the result of the *full multiplication*

$$\begin{array}{r|l} & 3x^2 \\ \hline 2x+1 & 6x^3 + 13x^2 + 13x + 7 \\ & -6x^3 - 3x^2 \\ \cline{2-2} \end{array}$$

Step III. We get the *first remainder* by oplussing the first opposite product

$$\begin{array}{r|l} & 3x^2 \\ \hline 2x+1 & 6x^3 + 13x^2 + 13x + 7 \\ & -6x^3 - 3x^2 \\ \cline{2-2} & \qquad 10x^2 + 13x \end{array}$$

Step IV. We decide if we want to stop or continue the division

– If we decide to *stop* the division,

  ∗ the *quotient* of the division is $+3x^{+2}$.

  ∗ the *remainder* of the division is $+10x^{+2} + 13x$

  If we don't care about the *remainder*, we write:

$$\frac{6x^3 + 13x^2 + 13x + 7}{2x + 1} = +3x^{+2} + (...)$$

  where we write $+ (...)$ as a reminder that $\dfrac{6x^3 + 13x^2 + 13x + 7}{2x + 1}$ is not exactly equal to $+3x^{+2}$ since there was a *remainder*.

– If we decide to *continue* the division, we start a new cycle

CYCLE 2. Step I. We find the *second monomial* in the quotient by *short division*:

$$\begin{array}{r|l} & 3x^2 \quad +5x \\ \hline 2x+1) & 6x^3 + 13x^2 + 13x + 7 \\ & -6x^3 \quad -3x^2 \\ \hline & \qquad 10x^2 + 13x \end{array}$$

Step II. We get the *second opposite product* by writing the opposite of the result of the *full multiplication*

$$\begin{array}{r|l} & 3x^2 \quad +5x \\ \hline 2x+1) & 6x^3 + 13x^2 + 13x + 7 \\ & -6x^3 \quad -3x^2 \\ \hline & \qquad 10x^2 + 13x \\ & \quad -10x^2 \quad -5x \\ \hline \end{array}$$

Step III. We get the *second remainder* by oplussing the fir second st opposite product

$$\begin{array}{r|l} & 3x^2 \quad +5x \\ \hline 2x+1) & 6x^3 + 13x^2 + 13x + 7 \\ & -6x^3 \quad -3x^2 \\ \hline & \qquad 10x^2 + 13x \\ & \quad -10x^2 \quad -5x \\ \hline & \qquad\qquad 8x + 7 \end{array}$$

Step IV. We decide if we want to stop or continue the division

– If we decide to *stop* the division,

  ∗ the *quotient* of the division is $+3x^{+2} + 5x$.

  ∗ the *remainder* of the division is $+8x + 7$

  If we don't care about the *remainder*, we write:

$$\frac{6x^3 + 13x^2 + 13x + 7}{2x + 1} = +3x^{+2} + 5x + (...)$$

where we write $+(\ldots)$ as a reminder that $\dfrac{6x^3+13x^2+13x+7}{2x+1}$ is not exactly equal to $+3x^{+2}+5x$ since there was a *remainder*.

– If we decide to *continue* the division, we start a new cycle

CYCLE 3. Step I. We find the *third monomial* in the quotient by *short division*:

$$
\begin{array}{r|l}
 & 3x^2 \;\; +5x+4 \\ \hline
2x+1 & 6x^3+13x^2+13x+7 \\
 & -6x^3 \;\; -3x^2 \\ \cline{2-2}
 & 10x^2+13x \\
 & -10x^2 \;\; -5x \\ \cline{2-2}
 & 8x+7
\end{array}
$$

Step II. We get the *third opposite product* by writing the opposite of the result of the *full multiplication*

$$
\begin{array}{r|l}
 & 3x^2 \;\; +5x+4 \\ \hline
2x+1 & 6x^3+13x^2+13x+7 \\
 & -6x^3 \;\; -3x^2 \\ \cline{2-2}
 & 10x^2+13x \\
 & -10x^2 \;\; -5x \\ \cline{2-2}
 & 8x+7 \\
 & -8x-4 \\ \cline{2-2}
\end{array}
$$

Step III. We get the *third remainder* by oplussing the third opposite product

$$
\begin{array}{r|l}
 & 3x^2 \;\; +5x+4 \\ \hline
2x+1 & 6x^3+13x^2+13x+7 \\
 & -6x^3 \;\; -3x^2 \\ \cline{2-2}
 & 10x^2+13x \\
 & -10x^2 \;\; -5x \\ \cline{2-2}
 & 8x+7 \\
 & -8x-4 \\ \cline{2-2}
 & 3
\end{array}
$$

Step IV. We decide if we want to stop or continue the division

– If we decide to *stop* the division,
  - ∗ the *quotient* of the division is $+3x^{+2}+5x+4$.
  - ∗ the *remainder* of the division is $+3$

  If we don't care about the *remainder*, we write:

$$\frac{6x^3+13x^2+13x+7}{2x+1} = +3x^{+2}+5x+4+(\ldots)$$

where we write $+(\ldots)$ as a reminder that $\dfrac{6x^3+13x^2+13x+7}{2x+1}$ is not exactly equal to $+3x^{+2}+5x+4$ since there was a *remainder*.

  - If we decide to *continue* the division, we start a new cycle
- When writing the partial remainders, we do not write the monomials beyond those that result from subtracting the *partial product*.

Example 14.6.

	$+4x$	$+3$		
$-3x^2 \ +5x \ -2$)	$-12x^3$	$+11x^2$	$-16x$	$+1$
First *opposite* partial product:	$+12x^3$	$-20x^2$	$+8x$	
First remainder:		$-9x^2$	$-8x$	
Second *opposite* partial product:		$+9x^2$	$-15x$	$+6$
Second remainder:			$-23x$	$+7$

The danger here is that, when we do the next subtraction, we may subtract from 0 rather than from the monomial that was unwritten in the partial remainder.

14.3 Division in Ascending Powers

One major difference between ARITHMETIC and POLYNOMIAL ALGEBRA is that:

- In ARITHMETIC, the *base* in the powers is always *larger* than ONE—in our case it is TEN but, for instance, COMPUTER SCIENCES use TWO, EIGHT and SIXTEEN as well.
- In POLYNOMIAL ALGEBRA, the base in the powers can be *smaller* than ONE as well as *larger* than ONE and, while this has no effect on the procedures we use for *addition*, *subtraction* and *multiplication*, whether x stands for numbers larger than 1 or smaller than 1 makes all the difference in the case of *division*.

This is because division usually does not stop by itself and *we* have to decide when to stop it. But we want to make sure that, after we have replaced the *unspecified numerator* by a *specific numerator*, what we kept of the quotient will give us *most* of what we should get so that we want the size of the successive results to go *diminishing*.

Now, as we already mentioned in Chapter 15, Section 2,

- When x is to be replaced by a numerator that is going to be *large in size*, we will want the Laurent polynomial to be written in *descending order of exponents*.
- When x is to be replaced by a numerator that is going to be *small in size*, we will want the Laurent polynomial to be written in *ascending order of exponents*.

As a result, we need to be able to divide in *ascending* order of exponents as well as in *descending* order of exponents. Fortunately, the procedure is exactly the same.

EXAMPLE 14.7. In order to compute $\dfrac{-12+23h-h^2-2h^3}{-3+2h}$, we divide $-3+2h$ into $-12+23h-h^2-2h^3$:

	$+4$	$-5h$	$-3h^2$	
$-3\ +2h$)	-12	$+23h$	$-h^2$	$-2h^3$
First *opposite* partial product:	$+12$	$-8h$		
First remainder:		$+15h$	$-h^2$	$-2h^3$
Second *opposite* partial product:		$-15h$	$+10h^2$	
Second remainder:			$+9h^2$	$-2h^3$
Third *opposite* partial product:			$-9h^2$	$+6h^3$
Third remainder:				$+4h^3$

- If we decide to *stop* the division,
  - the *quotient* of the division is $+4-5h-3h^2$.
  - the *remainder* of the division is $+4h^3$. Observe that if we replace the unspecified numerator h by, say, 0.2, then the remainder is equal to $4 \bullet 0.2^3 = 4 \bullet 0.008 = 0.032$ which is indeed small.

  If we don't care about the *remainder*, we write:

  $$\frac{-12+23h-h^2-2h^3}{-3+2h} = +4-5h-3h^2+(...)$$

  where we write $+\ (...)$ as a reminder that $\dfrac{-12+23h-h^2-2h^3}{-3+2h}$ is not exactly equal to $+4-5h-3h^2$ since there was a *remainder*.
- If we were to decide to *continue* the division, we would start a new cycle