Chapter 2

Signed Number-Phrases

There are two issues with plain number phrases:

• With plain numerators, we can count up as far as we want but we cannot always count down as far as we want.

• Plain number phrases can only represent collections in which all the items are of one same kind but there are many situations in which items can come in either one of two flavors.

**Example 2.1.**

- 3056.38 Dollars does not say if this was a deposit or a withdrawal,
- 37 800 Dollars does not say if a business is in the red (owes that money) or in the black (has that money).
- 62 Dollars does not say if a gambler is ahead of the game (has won more than s/he has lost) or in the hole (has lost more than s/he has won).
- 2 Feet from a benchmark does not say if a point is to the left or to the right of the benchmark.
- 5 Inches from the ground does not say if a point is above or below the ground.

2.1 Oriented Items

1. In the real world, there are many situations where we have to deal with collections of oriented items, that is items with either one of two orientations but where items with opposite orientations cancel each other so that collections of oriented items can only involve items that are all oriented the same way.
EXAMPLE 2.2. The collection \[
\begin{array}{c}
\rightarrow \rightarrow \rightarrow \rightarrow \\
\rightarrow \rightarrow \rightarrow \rightarrow \\
\rightarrow \rightarrow \rightarrow \rightarrow \\
\rightarrow \rightarrow \rightarrow \rightarrow \\
\end{array}
\] reduces automatically to only items with the same orientation: \[
\begin{array}{c}
\rightarrow \rightarrow \rightarrow \rightarrow \\
\rightarrow \rightarrow \rightarrow \rightarrow \\
\rightarrow \rightarrow \rightarrow \rightarrow \\
\rightarrow \rightarrow \rightarrow \rightarrow \\
\end{array}
\].

2. In the real-world, oriented items generally fall into either one of two categories:

- Items called **directed actions** which are “moves” of one kind or another but that can go either in **this-direction** or **that-direction**.

  **EXAMPLE 2.3.**
  - a businesswoman may *deposit* three thousand dollar on a bank account or may *withdraw* three thousand dollars from a bank account.
  - a gambler may *win* sixty-two dollars or may *lose* sixty-two dollars.
  - on a horizontal line, a point can be moved two feet *forward* or two feet *backward*.
  - on a vertical line, a point can be moved five inches *upward* or five inches *downward*.

- Items called **sided states** which can be either on **this-side** or **that-side** of some **benchmark**.

  **EXAMPLE 2.4.**
  - a business may be three thousand dollars *in the red* or three thousand dollars *in the black*.
  - a gambler may be sixty-two dollars *ahead of the game* or sixty-two dollars *in the hole*.
  - on a horizontal line with some benchmark, a point may be two feet *to the left* of the benchmark or two feet *to the right* of the benchmark.
  - on a vertical line with some baseline, a point may be five inches *above* the benchmark or five inches *below* the benchmark.
2.2 Signed Number Phrases

2.2.1 To make it clear which kind of numerator we are talking about, we will refer to the numerators which were introduced in ?? ?? as plain numerators.

To represent collections of oriented items on paper, we could of course just use plain numerators and two denominators, one for each orientation.

**Example 2.5.** We could represent the collection of oriented items \( \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \) by the plain number phrase 5 Left Arrows and the collection \( \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \) by the plain number phrase 5 Right Arrows.

However, even though orientation is arguably a qualitative matter, representing the orientation as part of the numerator instead of as part of the denominator will enormously facilitate computations.

1. **Signed numerators** consist of two parts:
   - The **size** of a signed numerator, which is its quantitative part, is the plain numerator that specifies “how many” or “how much”.
   - The **sign** of a signed numerator, which is its qualitative part, is the symbol, + or −, that specifies “which way”. The numerator 0 has no sign.

**Example 2.6.** Say the signed number-phrase \( +17.43 \text{ Dollars} \) specifies a money transaction. Then,

- **Size** \( +17.43 = 17.43 \) (plain number) specifies how much money was transacted,
- **Sign** \( +17.43 = + \) specifies which way the money went.

2. Then:

**Positive numerators** are the numerators whose sign is +.

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1Educologists will surely insist that we should conform. The reason we don’t use the term “absolute value” is that, while “−5 is-larger-in-size-than +2” is a relation in \( \mathbb{Z} \), \(|−5| > |+2| \) is a relation in \( \mathbb{N} \).
Negative numerators are the numerators whose sign is 

\[ \text{Example 2.7.} \quad \begin{array}{c}
\text{Positive Numerator: } +3 \\
\text{Negative Numerator: } -5
\end{array} \]

To make it always clear later what + is being used for, we will use

**Agreement 2.1.** In this text, to make it easy to distinguish positive numerators from plain numerators, the + sign will never “go without saying”.

**Example 2.8.** +51 is a positive numerator while 51 is a plain numerator.

3. The opposite of a signed numerator is the numerator with the same size and the other sign.

**Example 2.9.**

\[
\begin{array}{c}
\text{The opposite of } +3 \text{ is } -3 \\
\text{The opposite of } -5 \text{ is } +5
\end{array}
\]

If we want to write the opposite of a negative numerator which is a positive numerator, we must write the sign +, as otherwise, by Agreement 2.1, without the sign +, the numerator will be seen as being plain.

**Example 2.10.** If we want to talk about the opposite of \(-5\), we must write +5 because if we just write 5 this will be seen as a plain numerator which is not the opposite of anything and has no opposite.

4. In order to represent on paper a real-world collection of oriented items, the first thing we need to do is to declare:
   - which direction is to be represented by +. (And therefore which direction is to be represented by \(-\).)
   - which side of 0 is to be represented by +. (And therefore which side is to be represented by \(-\).)

**Example 2.11.** We declare that right steps are to be represented by +. (And therefore that that left steps are to be represented by \(-\).)
Then, to represent the collection \( \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \) we will use the signed number phrase \(-5\text{Arrows}\) and to represent the collection \( \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \) we will use the signed number phrase \(+5\text{Arrows}\).

**Example 2.12.** We declare that money *won* is to be represented by \(+\). (And therefore that money *lost* is to be represented by \(-\).)

When a *real-world* gambler: We write on *paper*:

- *wins* sixty-two dollars \(+62\text{ Dollars}\)
- *loses* sixty-two dollars \(-62\text{ Dollars}\)

in which \(+62\) is a *positive* signed-numerator and \(-62\) is a *negative* signed-numerator.

**Example 2.13.** We declare that accounts *in-the-black* are to be represented by *positive* numerators and that accounts *in-the-red* are to be represented by *negative* numerators and. Then,

When a *real-world* business is: We write on *paper*:

- three thousand dollars *in-the-black* \(+3000\text{ Dollars}\)
- three thousand dollars *in-the-red* \(-3000\text{ Dollars}\)

in which \(+3000\) is a *positive* signed-numerator and \(-3000\) is a *negative* signed-numerator.

### 2.3 Graphing Signed Number-Phrases

1. To *graph* signed number phrases, we use *signed number lines*.

**Example 2.14.** Here is a signed number line for signed numerators with \(\text{Arrows}\) as denominator:

\[
\begin{array}{cccccccc}
-4 & -3 & -2 & -1 & 0 & +1 & +2 & +3 & +4 & +5 \\
& & & & & & & & & \text{Arrows}
\end{array}
\]

Just as with plain number phrases, we will use *solid dots* and *hollow dots* to graph signed number phrases.

2. From the *graphic* viewpoint:
• The sign of a signed numerator codes which side of 0 the graph of the signed numerator is.

**Example 2.15.** Since
- Sign of $-5 = -$, the signed numerator $-5$ is left of 0.
- Sign of $+3 = +$, the signed numerator $+3$ is right of 0.
So the graphs are:

![Graph showing left and right of 0 with arrows]

• The size of a signed numerator codes how far away from 0 the signed numerator is on a signed number line.

**Example 2.16.** Since
- Size of $-5$ is 5, the signed numerator $-5$ is 5 away from 0.
- Size of $+5$ is 5, the signed numerator $+5$ is 5 away from 0.

![Graph showing 5 away from 0 with arrows]

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