Chapter 3

Comparisons. Equations - Inequations

3.1 Real-World Comparisons of Collections

Given two collections of the same kind of real-world items, a natural thing to do is to compare the first collection to the second collection.

1. Up front, there are three real-world relationships depending on, after we have completed the one-to-one matching, whether or not there are leftover items in one of the two collections:
   • When there are no leftover items, we will say that the first collection is equal to the second collection.

   **Example 3.1.** To compare Jack’s $100 with Jill’s $100 in the real world, we match Jack’s collection one-to-one with Jill’s collection:
Chapter 3. Comparisons

is not equal to
strict inequalities
is smaller than
is larger than

Since there is no leftover item in either collection, the relationship between Jack’s collection and Jill’s collection is that:

Jack’s collection is equal to Jill’s collection

• When there are leftover objects in either collection, we will say that the first collection is not equal to the second collection.

When two collections are not equal, there are two possible relationships, referred to as strict inequalities depending on which of the two collections the leftover item are in:

• When the leftover items are in the second collection, we will say that the first collection is smaller the second collection.

EXAMPLE 3.2. To compare Jack’s with Jill’s

in the real world, we match Jack’s collection one-to-one with Jill’s collection:

Since the leftover items are in Jill’s collection, the relationship between Jack’s collection and Jill’s collection is that:

Jack’s collection is smaller than Jill’s collection

• When the leftover objects are in the first collection, we will say that the first collection is larger than the second collection.

EXAMPLE 3.3. To compare Jack’s with Jill’s
3.2. Paper-World Comparison Sentences

in the real world, we match Jack’s collection one-to-one with Jill’s collection:

Since the leftover items are in Jack’s collection, the relationship between Jack’s collection and Jill’s collection is that:

Jack’s collection is larger than Jill’s collection

2. The three real-world relationships, is equal to, is smaller than, and is larger than, are mutually exclusive in the sense that as soon as we know that one of them holds, we automatically know that the other two relationships cannot hold. This brings about two other relationships referred to as weak inequalities, namely:

- no larger than which amounts to is smaller than or equal to,
- no smaller than which amounts to is larger than or equal to.

EXAMPLE 3.4. If we know that a first collection is no larger than a second collection, then this means that the first collection is either smaller than the second collection or equal to the second collection!

3.2 Paper-World Comparison Sentences

We now expand our mathematical language with verbs to represent on paper the above real-world relationships.

1. To represent on paper
   - Real-world equalities:
     - is smaller than: We will use the verb < which is read is less than,
   - Real-world strict inequalities:
     - is smaller than: We will use the verb < which is read is less than,
     - is larger than: We will use the verb > which is read is greater than,
   - Real-world weak inequalities:
     - is no larger than: We will use the verb \leq \) which is read less than or equal to,
Chapter 3. Comparisons

- *is no smaller than*: We will use the verb $\geq$ which is read *greater than or equal to*.

2. Then, to indicate that a relationship *holds* from one collection to another, we will write a **comparison sentence** that consists of the *number-phrases* that represent the two collections with the *verb* that represents the *relationship* in-between the two number-phrases.

**Example 3.5.** Given Jack’s $\text{\$3}$ and Jill’s $\text{\$3}$, we represent the real-world *relationship* 

Jack’s collection *is equal to* Jill’s collection

by writing on paper the *equality*

$$3 \text{ Washingtons} = 3 \text{ Washingtons}$$

which we read as

**three Washingtons is equal to three Washingtons.**

**Example 3.6.** Given Jack’s $\text{\$3}$ and Jill’s $\text{\$7}$, we represent the real-world *relationship* 

Jack’s collection *is smaller than* Jill’s collection

by writing the *strict inequality*

$$3 \text{ Washingtons} < 7 \text{ Washingtons}$$

which we read as

**three Washingtons is less than seven Washingtons.**

**Example 3.7.** Given Jack’s $\text{\$5}$ and Jill’s $\text{\$3}$, we represent the real-world *relationship* 

Jack’s collection *is larger than* Jill’s collection

by writing the *strict inequality*

$$5 \text{ Washingtons} > 3 \text{ Washingtons}$$

which we read as

**five Washingtons is greater than three Washingtons.**
3. There are two very different ways we can compare signed numerators.

a. We can compare the signed numerators themselves. Then, on a number line, the signed numerator that is smaller is the one to the left and the signed numerator that is larger is the one to the right.

**Example 3.8.** Given the signed numerators $-7$ and $-1$, we have $-7 < -1$. and, graphically, we have:

```
<table>
<thead>
<tr>
<th>Liters</th>
</tr>
</thead>
<tbody>
<tr>
<td>-8.</td>
</tr>
<tr>
<td>-7.</td>
</tr>
<tr>
<td>-6.</td>
</tr>
<tr>
<td>-5.</td>
</tr>
<tr>
<td>-4.</td>
</tr>
<tr>
<td>-3.</td>
</tr>
<tr>
<td>-2.</td>
</tr>
<tr>
<td>-1.</td>
</tr>
<tr>
<td>0.</td>
</tr>
<tr>
<td>+1.</td>
</tr>
<tr>
<td>+2.</td>
</tr>
<tr>
<td>+3.</td>
</tr>
<tr>
<td>+4.</td>
</tr>
<tr>
<td>+5.</td>
</tr>
<tr>
<td>+6.</td>
</tr>
</tbody>
</table>
```

b. But we can also compare signed numerators according to their size. Then, on a number line, the signed numerator that is smaller-in-size is the one closer to 0 and the signed numerator that is larger-in-size is the one farther from 0.

**Example 3.9.** Given the signed numerators $-6$ and $+3$, we have

```
<table>
<thead>
<tr>
<th>Liters</th>
</tr>
</thead>
<tbody>
<tr>
<td>-8.</td>
</tr>
<tr>
<td>-7.</td>
</tr>
<tr>
<td>-6.</td>
</tr>
<tr>
<td>-5.</td>
</tr>
<tr>
<td>-4.</td>
</tr>
<tr>
<td>-3.</td>
</tr>
<tr>
<td>-2.</td>
</tr>
<tr>
<td>-1.</td>
</tr>
<tr>
<td>0.</td>
</tr>
<tr>
<td>+1.</td>
</tr>
<tr>
<td>+2.</td>
</tr>
<tr>
<td>+3.</td>
</tr>
<tr>
<td>+4.</td>
</tr>
<tr>
<td>+5.</td>
</tr>
<tr>
<td>+6.</td>
</tr>
</tbody>
</table>
```

and so, since $-6$ is farther from 0, $-6$ is larger-in-size and since $+3$ is closer to 0, $+3$ is smaller-in-size.

**Language 3.1. Comparison symbols** The symbols $<, \leq, >, \geq$, and $=$ all refer to the comparison of the signed numerators themselves. There are no symbols for the comparison of signed numerators according to their size.
3.3 TRUE Versus FALSE

Inasmuch as the *comparison-sentences* that we wrote until now represented relationships between given *real-world* collections, they were TRUE.

1. However, there is nothing to prevent us from *writing* comparison-sentences that are FALSE in the sense that there is no way anyone could come up with real-world collections for which *one-to-one matching* would result in the relationships represented by these comparison-sentences.

**EXAMPLE 3.10.** The sentence

\[ 5 \text{ Washingtons} = 3 \text{ Washingtons}, \]

is FALSE because there is no way anyone could come up with real-world collections that could be represented by 5 *Washingtons* and 3 *Washingtons* and for which the *one-to-one matching* would result in there being no leftover item.

**EXAMPLE 3.11.** The sentence

\[ 5 \text{ Washingtons} < 3 \text{ Washingtons} \]

is FALSE because there is no way anyone could come up with real-world collections that could be represented by 5 *Washingtons* and 3 *Washingtons* and for which the *one-to-one matching* would result in there being leftover items in the second collection.

**EXAMPLE 3.12.** The sentence

\[ 3 \text{ Washingtons} > 5 \text{ Washingtons}, \]

is FALSE because there is no way anyone could come up with real-world collections that could be represented by 3 *Washingtons* and 5 *Washingtons* and for which the *one-to-one matching* would result in there being leftover items in the first collection.

2. But, while occasionally useful, it is usually not very convenient to write sentences that are FALSE because then we must remember also to indicate that they are FALSE. So, inasmuch as possible, we shall write only sentences that are TRUE and we will use

**AGREEMENT 3.1. (Truth)** When no indication of truth or falsehood is given, comparison sentences will be understood to be TRUE and this will go without saying.
3.4 Comparing Number-Phrases

But even when we want to write a comparison sentence that is FALSE we don’t want having to remember to say that it is FALSE, so what we will do is to write the negation of the comparing sentence—which is therefore TRUE which therefore “goes without saying”. To write the negation of a given sentence, we can proceed in either one of two ways:

- We can enclose the comparison sentence that is FALSE in square brackets [ ] and prefix the whole thing with the symbol NOT,

  **Example 3.13.** Instead of writing:
  The sentence \( 5 \text{ Washingtons} \equiv 3 \text{ Washingtons} \) is FALSE
  we can write the sentence
  \[ \text{NOT}[5 \text{ Washingtons} = 3 \text{ Washingtons}] \]

  or

- We can just slash the verb,

  **Example 3.14.** Instead of writing:
  The sentence \( 5 \text{ Washingtons} \equiv 3 \text{ Washingtons} \) is FALSE
  we can write the sentence
  \( 5 \text{ Washingtons} \neq 3 \text{ Washingtons} \)

- With strict inequalities, we can also use the “other” weak inequality.

  **Example 3.15.** Instead of writing that
  The sentence \( 5 \text{ Washingtons} < 3 \text{ Washingtons} \) is FALSE
  we can write the sentence
  \( 5 \text{ Washingtons} \preceq 3 \text{ Washingtons} \)

  or we can write the sentence
  \( 5 \text{ Washingtons} \succeq 3 \text{ Washingtons} \)

3.4 Comparing Number-Phrases

1. To compare two counting number-phrases, we must see whether we must count-up or count-down from the first numerator to the second numerator\(^1\).

---

\(^1\)Educologists will be glad to know that, already in 1905, Fine was deliberately using the cardinal aspect for comparison processes in the real world and the ordinal aspect for comparison procedures on paper.
Chapter 3. Comparisons

- If we must count up, then the comparison-sentence is the strict inequality:
  
  \[
  \text{first counting number-phrase} \prec \text{second counting number-phrase}
  \]
  
  (with \(\prec\) read as “is-less-than”)

**Example 3.16.** To compare the counting number-phrases

3 Washingtons and 7 Washingtons

i. We must count from 3 to 7 that is we must count \(\text{up}\):

4, 5, 6, 7

ii. So, we write the **strict inequality**:

3 Washingtons \(\prec\) 7 Washingtons

- If we must count down, then the comparison-sentence is the strict inequality:

  \[
  \text{first counting number-phrase} \succ \text{second counting number-phrase}
  \]
  
  (with \(\succ\) read as “is-more-than”)

**Example 3.17.** To compare the counting number-phrases

9 Washingtons and 2 Washingtons

i. We must count from 9 to 2 that is, we must count \(\text{down}\):

2, 3, 4, 5, 6, 7, 8

or

8, 7, 6, 5, 4, 3, 2

ii. So, we write the **strict inequality**:

9 Washingtons \(\succ\) 2 Washingtons

- If we must neither count up nor count down, then the comparison-sentence is the equality:

  \[
  \text{first counting number-phrase} = \text{second counting number-phrase}
  \]
  
  (with \(=\) read as “is-equal-to”)

**Example 3.18.** To compare the counting number-phrases

3 Washingtons and 3 Washingtons.

i. We must count from 3 to 3, that is we must count neither \(\text{up}\) nor \(\text{down}\).

ii. So, we write the **equality**:

3 Washingtons = 3 Washingtons
2. A **basic problem** is a problem in which the number-phrases we want must:
- Come out of a given set of number-phrases called the **data set**
- Compare with a given **gauge number-phrase** in a way given by a **comparison form**

The **solution subset** of the problem then consists of all the **solutions**, that is of all the number-phrases in the **data set** that turn the **comparison form** into a **true comparison sentence**.

**Example 3.19.** Given the **problem** in which:
- the **data set** is \{3 \text{Washington}, 12 \text{Washington}, 15 \text{Washington}, 21 \text{Washington}\}
- the **comparison form** is \(\underline{} > 10 \text{Washington}\)
  (where 10 \text{Washington} is the given **gauge collection**.)

Since we have:
- \(3 \text{Washington} > 12 \text{Washington}\) is **false**
- \(12 \text{Washington} > 12 \text{Washington}\) is **false**
- \(15 \text{Washington} > 12 \text{Washington}\) is **true**
- \(21 \text{Washington} > 12 \text{Washington}\) is **true**

the **solution subset** is \{15 \text{Washington}, 21 \text{Washington}\}

3. We will avoid writing all these denominator by:
- **declaring** up-front the **denominator** common to all the number-phrases, both in the data set and in the comparison form,
- **writing only the numerators**, both in the data set and the specifying form and thereafter.

**Example 3.20.** Instead of writing the basic problem
- \{3 \text{Washington}, 12 \text{Washington}, 15 \text{Washington}, 21 \text{Washington}\} (the **data set**)
- \(\underline{} > 12 \text{Washington}\) (the **specifying form**)

we **declare** that the problem is in **Washington** and we only write:
- \{3, 12, 15, 21\} (the **data set**)
- \(\underline{} > 12\) (the **specifying form**)

and then:

Since we have
- \(3 > 12\) is **false**
- \(12 > 12\) is **false**
- \(15 > 12\) is **true**
- \(21 > 12\) is **true**
Chapter 3. Comparisons

the solution subset is \{15, 21\}

4. In algebra, though, instead of using a comparison form with a box in which we can write-in number-phrases, we will use a comparison formula (We will often just say formula for short) with an unspecified-numerator such as, for instance, the letter \(x\) for which we can substitute numerators.

**Example 3.21.** Instead of writing the specifying form

\[
\text{[ ]} < 5
\]

we will write the specifying formula

\[x < 5\]

5. In a basic problem, the comparison verb can be any one of:

\[=, \neq, \leq, \geq, >, <.\]

**Example 3.22.** The following are basic comparison-formulas

\[
x = 12
\]
\[
x \neq 12
\]
\[
x < 8
\]
\[
x > 23
\]
\[
x \leq 8
\]
\[
x \geq 8
\]

3.5 Basic Equation Problems

1. When the data set consists of counting numerators we proceed as follows:

- If there is a numerator in the data set which is the same as the gauge numerator, then the solution subset consists of just that numerator.
- If none of the numerators in the data set is the same as the gauge numerator, then the solution subset is empty.

**Example 3.23.** Given the basic equation problem
In order to solve this basic problem,

i. There is a numerator in the data set which is the same as the gauge numerator, namely 7:

ii. We write the name of the solution subset: \( \{7\} \text{ Apples} \).

iii. We graph the solution subset:

2. When the data set consists of all real numerators, since the gauge numerator is a real numerator, it is necessarily in the solution subset.
Chapter 3. Comparisons

**EXAMPLE 3.25.** Given the basic inequation problem

1. **Data set:** \{ All plain real numerators \} Meters
2. **Comparison formula:** \( x = 32.67 + [...] \)

i. Since the gauge numerator \( 32.67 + [...] \) is a real numerator, it is automatically in the solution subset

ii. We write the name of the solution subset: \{ \( 32.67 + [...] \) \} Meters.

iii. We graph the solution subset:

\[ x \geq 32.67 + [...] \]

### 3.6 Basic Inequation Problems

1. When the data set consists of counting numerators,
   i. We declare \( x \) to be each and every numerator in the data set, write the resulting comparison sentence and write TRUE or FALSE.
   
   ii. We write the name of the solution subset
   
   iii. We graph the solution subset.

**EXAMPLE 3.26.**

1. **Data set:** \{ 2, 3, 6, 7, 9 \} Apples
2. **Comparison formula:** \( x \leq 7 \)

   i. Numerator Set
   
   ii. Common Denominator
   
   iii. Unspecified numerator
   
   iv. Gauge numerator

\[ x \leq 7 \]
3.6. Basic Inequation Problems

i. We declare \( x \) to be each and every numerator in the data set:
- \( x \leq 7 \) gives the comparison sentence \( 2 \leq 7 \) which is \text{TRUE}
- \( x \leq 7 \) gives the comparison sentence \( 3 \leq 7 \) which is \text{TRUE}
- \( x \leq 7 \) gives the comparison sentence \( 6 \leq 7 \) which is \text{TRUE}
- \( x \leq 7 \) gives the comparison sentence \( 7 \leq 7 \) which is \text{TRUE}
- \( x \leq 7 \) gives the comparison sentence \( 9 \leq 7 \) which is \text{FALSE}

ii. We write the name of the solution subset: \( \{2, 3, 6, 7\} \) Apples.

iii. We graph the solution subset:

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array}
\]

\text{Apples}

2. When the data set consists of all plain real numerators, we use the Pasch Procedure:

**Example 3.27.**

Given the basic inequation problem

- **Data set:** \{All plain real numerators\} Numerator Set
- **Comparison formula:** \( x \) \( > \) \( 32.67 + [...] \) Gauge numerator

We get the solution subset using the Pasch Procedure:

**1. Locating the Boundaries:**

i. Solving the associated equation problem

\[ x = +32.67 + [...] \]

gives the boundary \( +32.67 + [...] \).

ii. To check if the boundary is in the solution subset we declare \( x \) to be \( +32.67 + [...] \) in the given inequation:

\[ x > +32.67 + [...] \]

\[ x \xrightarrow{\text{ineq}} +32.67 + [...] \]
that is
\[ +32.67 + [...] > +32.67 + [...] \]
which is false so that \[ +32.67 + [...] \] is not in the solution subset and we graph the boundary of the solution subset \[ +32.67 + [...] \] with a hollow dot:

2. Locating the Interior: We test each one of the two intervals that the boundary divides the data set into:

Interval A

Interval B

\[ +32.67 + [...] \]

\[ +32.67 + [...] \]

- To test Interval A, we declare \( x \) to be \(-1000\) in the given inequation:

\[ x > +32.67 + [...], x = -1000 \]

that is

\[ -1000 > +32.67 + [...] \]

which is false so that \(-1000\) is not in the solution subset and by the Pasch Theorem the whole Interval A is not part of the solution subset and therefore we graph Interval A with a hollow bar:

Interval A

\[ +32.67 + [...] \]

- To test Interval B, we declare \( x \) to be \(+1000\) in the given inequation:

\[ x > +32.67 + [...], x = +1000 \]

that is

\[ +1000 > +32.67 + [...] \]

which is true so that \(+1000\) is in the solution subset and by the Pasch Theorem the whole Interval B is part of the solution subset and therefore we graph Interval B with a solid bar:
3. Altogether:
   - The teacher’s graph of the solution subset is:
   
   ![Diagram of interval B in dollars]

   - The name of the solution subset is:
     
     $\left[ +32.67 + [...], +\infty \right)$ Dollars
Index

\(<\), 3
\(\geq\), 3, 4
\([\,]\), 7
\(x\), 10
FALSE, 6
NOT, 7
TRUE, 6

basic problem, 9
compare, 1, 5
comparison form, 9
comparison formula, 10
comparison sentence, 4
data set, 9
declare, 9
empty, 10
formula, 10
gauge number-phrase, 9
hold, 3

is equal to, 1
is larger than, 2
is not equal to, 2
is smaller than, 2

larger, 5
larger-in-size, 5
leftover item, 1

mutually exclusive, 3
negation, 7
no larger than, 3
no smaller than, 3
prefix, 7
slash, 7
smaller, 5
smaller-in-size, 5
solution subset, 9
solutions, 9
square brackets, 7
strict inequalities, 2
substitute, 10

unspecified-numerator, 10
verb, 3
weak inequalities, 3