Chapter 1

Symbolic Systems

What is important is the real-world, that is physics, but it can be explained only in mathematical terms.

Dennis Serre

Contrary to what most people think, Mathematics in general, and Arithmetic in particular, originate in the real-world and deal with real-world entities and the relationships among real-world entities. However, in order to:

• Communicate about the real-world, that is share information about the real world,
• Investigate how the real-world works,
• Plan in advance our actions on the real-world (because acting on the real-world without thinking ahead usually has very bad consequences),

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we need a **language** to **represent** the real world on **paper**.

And, indeed, there are all sorts of languages: anywhere between 3000 and 8000 spoken languages, hundreds of sign-languages, etc. There are written languages, pictorial languages, secret and non-secret codes, bar codes, computer languages, etc. (See [http://en.wikipedia.org/wiki/Language](http://en.wikipedia.org/wiki/Language).)

Nevertheless, as we will see, **MATHEMATICS** requires a language of its own.

### 1.1 What Languages Are Made Of

Very, very roughly, most languages are constructed as follows:

1. The building blocks of the language are **words** listed in a **vocabulary** that must be available to all users of the language.

**Example 1.1.1.** Of the following:

- **word**, wood
- **woord**, wod

the first two are English words but the other two are not.

The two main kinds of **words** that we will be using are:

- **Nouns** to represent real-world **entities**,
- **Verbs** to represent **relationships** among real-word entities.

**Example 1.1.2.** Of the following:

- **apple**, triangle
- **eats**, intersects

the first two are **nouns** and the other two are **verbs**.

2. Quite often, though, we do not have single ready-made words to represent real-word entities or to represent a relationship among real-word entities and we will have to use a **phrase**, that is “a small group of words standing together as a conceptual unit” ([http://www.oxforddictionaries.com/us/definition/american_english/phrase](http://www.oxforddictionaries.com/us/definition/american_english/phrase)). More precisely, we will use:

- **Noun phrases**, that is groups of words that work the same way single nouns do, to represent real-world entities,
- **Verb phrases**, that is groups of words that work the same way single verbs do, to represent relationships among real-word entities.

We will place **hyphens** between the words that make up a phrase to emphasize the fact that these words are to be read together as a whole.

**Example 1.1.3.** Of the following:

- **ten-dollar-bill**, mathematics-teacher
- **takes-apart**, is-the-daughter-of

the first two are **noun phrases** and the other two are **verb phrases**.
3. In order to communicate we use **sentences**, that is groups of **words** or **phrases** constructed according to **grammatical rules** listed in a **grammar** that must be available to all users of the language.

**Example 1.1.4.** Of the following:
- The hamburger paints the ocean with the mountains.
- Dog fox lazy quick brown the jumps the over.
the first is an English **sentence** but the second is not an English sentence.

4. However, a **sentence** may or may not be **stating** something about the real-world. A **sentence** that states something about the real-world is called a **statement** and what it states about the real-world is called the **meaning** of the sentence. A sentence that does not state something about the real-world is said to be **meaningless**.

**Example 1.1.5.** Of the following two **sentences**:
- The spaghetti sauce is too hot to be eaten with a spoon.
- The rose understands the navy with a table.
the first is a **statement** (about the real-world) but the second is **meaningless** since it is not a statement (about the real-world).

5. A sentence is then:
- **TRUE** if what it states about the real-world **occurs** in the real-world,
- **FALSE** if what it states about the real-world does **not** occur in the real-world.

**Example 1.1.6.** The following are both (famous) **statements** about the real-world:
- The moon is made of green cheese.
- Humans are the only featherless bipeds.
However,

The first statement is **false** because, as Armstrong checked on July 20, 1969, the moon is **not** made of green cheese.

while

The second statement is **true** because, even though there are many species that are featherless and many species that use only two legs for walking, it so happens that humans are the only ones to be both featherless and bipedal.

6. Our main goal in this text will be, using **sentences**, to develop **procedures** to simulate on paper real-world **processes**.

**Example 1.1.7.** Suppose we have to carry out the real-world **process** of handing out seventeen dollars to each one of two thousand six hundred and forty eight people. A preliminary issue then is whether we have the cash to carry out this real-world process. To find out, we **represent** on paper both how many dollars per person and how many persons and we then use the paper-world **procedure** called **multiplication** to figure out on paper how many dollars we will need in order to carry out the **process** in the real world.
1.2 Learning Languages

Learning a language is nowhere as simple as one might think.

1. The language that each one of us first learned as a child, our native language, we learned by directly associating nouns with the real-world entities that the nouns represent. But how children learn verbs to represent real-world relationships among real-world entities still isn’t fully understood.

2. A second language is usually not learned the same way as a native language, that is not by direct association with the real-world, but through the language that we already know. So, learning a second language automatically involves two languages which we will need to keep separate:
   - The object language, which is the language that we want to learn,
   - The metalanguage, which is the language we already know and which we use to learn the object language.

Example 1.1.8. For an American learning Spanish, the object language is Spanish and English is the metalanguage but for a Spaniard learning English, the object language is English and Spanish is the metalanguage.

3. In a book, the metalanguage is used for several different purposes:
   - To describe and discuss the real-world to be to be represented on paper with the object language. In particular, we will use the metalanguage to describe and discuss:
     - The real-world entities to be represented on paper,
     - The real-world relationships among real-world entities to be represented on paper;
   - but also
     - To describe and discuss features of the object language itself,
   - and, more generally,
   - To discuss matters.

1.3 In This Book

We will be using English as our metalanguage.

1. Since we will not be able to exhibit the real-world items that we will want to represent on paper, we will usually use pictures as substitutes for the real-world entities. However, as even using pictures will not always be possible, we will also use as substitutes English words which will then be printed in bold-faced italics. So, in this book, words printed in bold-faced italics will belong to the metalanguage.
1.4 Natural Languages

**EXAMPLE 1.1.9.** Say we intend to discuss the paper-representation of *real-world* one-dollar bills. Since we cannot exhibit these *real-world* one-dollar bills, we will use *pictures* as a *substitute* for these real-world one-dollar bills, for instance

However we will also use as *substitutes* for these real-world one-dollar bills the English words (in bold-faced italics),

```
one-dollar bill  one-dollar bill  one-dollar bill
```

**NOTE.** It is very important to distinguish *nouns* from *substitutes*:

- *nouns* are intended to *represent on paper* the real-world items.

while

- *substitutes* are intended to *stand for* the real-world items and will be used only when we cannot use *pictures* to do so.

2. When there will be too many real-world entities for us to use pictures or English words as substitutes, we will use English numbers in *small caps* as a *shorthand*. So, in this book, numbers spelled out in small caps belong to the metalanguage.

**EXAMPLE 1.1.10.** We will use *five apples* as a shorthand for *apple apple apple apple apple* as well as *five ●* as a shorthand for ●●●●●.

3. In order to make it easier to find in this book the place where words, both in the object language and in the metalanguage, appear for the first time and are explained, this book uses a standard method: These words are

- *boldfaced* the first time they appear in this book and are explained,
- printed in the *margin* of the page where they appear for the first time and are explained,
- listed in the *index* at the end of the book along with the number of the page where they appear for the first time and are explained.

1.4 Natural Languages

A *natural language* is the kind of language we use most of the time, routinely, and rather loosely. Natural languages depend on place and time.

**EXAMPLE 1.1.11.**
specialized languages

The real-world entity whose *picture* is

![Table](image)

is represented by different *words* depending on the natural language we are using:

<table>
<thead>
<tr>
<th>Language</th>
<th>Word</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chinese</td>
<td>表</td>
</tr>
<tr>
<td>English</td>
<td>table</td>
</tr>
<tr>
<td>Finnish</td>
<td>taulukko</td>
</tr>
<tr>
<td>French</td>
<td>table</td>
</tr>
<tr>
<td>German</td>
<td>Tisch</td>
</tr>
<tr>
<td>Greek</td>
<td>τραπέζι</td>
</tr>
<tr>
<td>Italian</td>
<td>tavola</td>
</tr>
<tr>
<td>Japanese</td>
<td>食卓</td>
</tr>
<tr>
<td>Latin</td>
<td>tabula</td>
</tr>
<tr>
<td>Russian</td>
<td>стул</td>
</tr>
<tr>
<td>Spanish</td>
<td>mesa</td>
</tr>
<tr>
<td>Vietnamese</td>
<td>bàn</td>
</tr>
</tbody>
</table>

How natural languages were born and how they then evolved and interacted with one another is buried in time and investigated in *Etymology* (See [http://en.wikipedia.org/wiki/Etymology](http://en.wikipedia.org/wiki/Etymology)).

### 1.5 Specialized Languages

While natural languages usually serve us tolerably well for general communication, they have severe limitations.

1. People working in any trade need *specialized languages* because:
   - Workers need to know exactly what they are to do.

   **EXAMPLE 1.1.12.** We cannot just tell an architect “Design me a building”. Architects need *specifications*. In turn, architects cannot just give pictures to construction workers. Construction workers need at least *blueprints*.

   - Workers need to communicate with each other while doing the work.

   **EXAMPLE 1.1.13.** Imagine the likely consequences of two persons working with some dangerous machinery and one telling the other: “Push the gizmo which is on the whatchamacallit next to the doodad on the other side of the doohickey.”

2. Sometimes, specialized languages use made-up words but most of the time they use common words with a meaning special to the trade.

   **EXAMPLE 1.1.14.** When an *electrician* asks for a pancake, that’s not because s/he is hungry but because s/he needs a junction box the thickness of drywall. When a *carpenter* asks for *drywall*, s/he is not likely to be asking for a wall that is not wet. When a *mason* asks for a *hawk*, s/he is not likely to be asking for a bird.

   **EXAMPLE 1.1.15.** “[The actor] must learn the theatre’s special vocabulary. Partly technical, partly slang, much of it is standardized on the English-speaking stage. As a working actor, you must be familiar with this language, just as a mechanic must know the names of his tools or a surgeon the names of her instruments.” (From *Acting is Believing* by Charles McGaw, Kenneth L. Stilson, Larry D. Clark.)
1.6. Mathematical Language

**Example 1.1.16.** "If you look up the meaning of the word "theory" in the dictionary, it is described as being a synonym for words like "proposition", "hypothesis", or even "speculation". In contrast, a scientific theory is an established body of knowledge about a certain subject, supported by observable facts, repeatable experiments, and logical reasoning. A theory in science is a formal explanation of some aspect of the natural world, tested and verified by careful observation and experimentation. A good theory is one that also produces accurate and useful predictions." (From https://plus.maths.org/content/evolution-its-real-gravity?nl=0.)

1.6 Mathematical Language

In the mathematical trade, the situation is exactly the same as in any other trade except even more so because, in mathematics, just as if we were in a court of law, we always have to make-the-case that:

- The **mathematical sentences** we write are **true**, that is are statements about the real-world whose meaning occurs in the real-world,
- The **mathematical procedures** we develop are **correct**, that is simulate the real-world processes.

1. As a result, we will have to develop an object language that makes it as easy as possible for us to compute, that is to manipulate the words in the object language according to listed deductive rules. So, the main feature we will require from the object language is that it be **systematic**.

2. In order to stress the systematic aspect of the object language that we will develop, we will often use **symbols** instead of **words** and we will use the phrase **symbolic-system** instead of the word **language**. More precisely, a symbolic-system is specified by a vocabulary of symbols:
   - **Noun-symbols** to represent real-world entities,
   - **Verb-symbols** to represent relationships among real-world entities, together with
   - **Logical-symbols** to represent in the symbolic-system the meaning of the English words “and”, “or”, “not”. We will use &, ∨, ¬ along with a **grammar**.

3. We can then assemble these symbols to write **symbolic-sentences**. These symbolic sentences will then be **true** or **false** depending on whether they make statements about the real-world whose meaning occurs or does

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2Educologists will surely object to our excluding “if ... then ...”. The reason is that, instead of using implication (a binary operation among sentences), we find it more natural to say that a sentence yields another sentence (a binary relation among sentences) to represent on paper logical consequence.
not occur in the real-world. We will then say that we have coded the information about the real-world into the symbolic-system.

**Example 1.1.17.** Let’s say we want to code the following real-world situation:

- The entities are three persons: Andy, Beth and Cathy.
- The relationship is loves/hates as determined by the following table:

<table>
<thead>
<tr>
<th>loves/hates</th>
<th>Andy</th>
<th>Beth</th>
<th>Cathy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Andy</td>
<td>loves</td>
<td>loves</td>
<td>loves</td>
</tr>
<tr>
<td>Beth</td>
<td>loves</td>
<td>hates</td>
<td>loves</td>
</tr>
<tr>
<td>Cathy</td>
<td>hates</td>
<td>loves</td>
<td>hates</td>
</tr>
</tbody>
</table>

So, for instance, in the real-world according to the above table, we have that Andy loves Cathy, Cathy hates Andy, etc.

i. In order to code the information about this real-world situation, we may, for instance, use the following symbolic-system:

- The vocabulary consists of the following symbols:
  - Noun-symbols: $a$ to represent Andy, $b$ to represent Beth, $c$ to represent Cathy.
  - Verb-symbol: $L$ to represent loves,
  - Logical-symbol: $\&$ to represent and.
- The grammar says that the noun-symbols should be on either sides of the verb-symbol—just as in English. (But, for another possible grammar, see [http://en.wikipedia.org/wiki/Reverse_Polish_notation](http://en.wikipedia.org/wiki/Reverse_Polish_notation).)

ii. Then, for instance:

- The symbolic-sentence $bLa$ means Beth loves Andy and since, in the real-world according to the above table, Beth loves Andy, the symbolic-sentence $bLa$ is true.

- The symbolic-sentence $cLa$ means Cathy loves Andy and since, in the real-world according to the above table, Cathy hates Andy, the symbolic-sentence $cLa$ is false.

- The symbolic-sentence $cLb \& cLa$ means that Cathy loves Beth and Cathy loves Andy and since, in the real-world according to the above table, Cathy loves Beth but Cathy hates Andy, the symbolic-sentence $cLb \& cLa$ is false.

**Note.** In this symbolic-system, we would not be able to write a symbolic-sentence to mean that Cathy loves Beth or Cathy loves Andy because the vocabulary in this symbolic-system does not include a logical-symbol to represent or.

4. Aside from the need to compute with symbols, we will also want to be able to make comments in the metalanguage in order to describe, explain and discuss what it is that we are doing within the symbolic-system.
**Example 1.1.18.** Software engineers will tell you that any computer code *must* include comments about what the code is supposed to do that are intended to be read by humans—while being ignored by the computer. Indeed, in the absence of comments, the code rapidly becomes unusable because after a while the way it works cannot be understood anymore and therefore the code cannot be maintained. So, every computer language includes a way for there to be comments.

5. Finally, we have to agree to write in the *standard* mathematical language. This is because we are trying to write sentences that are true and the only way to make sure that our sentences are true is to submit to others the case we made for their truth. And of course, if we do not write in the standard mathematical language, nobody is going to look at, even less debate, our case and so we will never know if the sentences we wrote are true or false. See [http://en.wikipedia.org/wiki/Peer_review](http://en.wikipedia.org/wiki/Peer_review)

However, mathematicians do allow deviations from and additions to the standard mathematical language because:

i. A symbolic-system for dealing with all of mathematics would be so big as to be totally unusable,

and so,

ii. Given a particular purpose, parts of the *standard* mathematical language have to be re-cycled and, more generally, deviations are often unavoidable.

Of course, as in computer programming, any deviation from and/or addition to the standard mathematical language has to be declared up front. And a look at any real mathematical text will show how often mathematicians use the word *definition*.

### 1.7 Symbolic Systems for Arithmetic

As opposed to **Geometry** which deals with the *shape* of entities, **Kinematics** which deals with the *speed* of entities, **Dynamics** which deals with the way entities react to *forces*, **Thermodynamics** which deals with the temperature of entities, etc, **Arithmetic** deals with *quantities* of entities.

1. What immediately complicates matters, though, is that, there are two very different kinds of real world *entities* and, correspondingly, two very different kinds of *quantities* ³:

³Here, Educologists would of course speak of *multitude* versus *magnitude*. The trouble, though, is that both terms usually connote “big”.
Chapter 1. Symbolic Systems

• Items are distinct entities we can deal with individually and instead of speaking of a quantity of items we speak of a number of items.

**Example 1.1.19.** Apples are items we can deal with one at a time and we speak of a number of apples.

• A substance is an entity we can deal with only in bulk and instead of speaking of a quantity of substance we speak of an amount of substance.

**Example 1.1.20.** Soup is a substance we can deal with only in bulk and we speak of an amount of soup.

2. Natural languages often use different words when dealing with numbers and when dealing with amounts but not always.

**Example 1.1.21.** In English, the distinction between numbers and amounts is not always made:

<table>
<thead>
<tr>
<th>Numbers</th>
<th>Amounts</th>
</tr>
</thead>
<tbody>
<tr>
<td>Too many apples</td>
<td>Too much soup</td>
</tr>
<tr>
<td>A few apples</td>
<td>A little soup</td>
</tr>
<tr>
<td>Fewer apples than bananas</td>
<td>Less soup than milk</td>
</tr>
<tr>
<td>More apples than bananas</td>
<td>More soup than milk</td>
</tr>
<tr>
<td>Many more apples than bananas</td>
<td>Much more soup than milk</td>
</tr>
</tbody>
</table>

3. As we will see, the distinction between numbers and amounts is an absolutely fundamental one, especially in sciences and technologies. (See [http://www.differencebetween.net/science/mathematics-statistics/difference-between-number-and-amount/](http://www.differencebetween.net/science/mathematics-statistics/difference-between-number-and-amount/) and [http://www.youtube.com/watch?v=F8Do9bDfd1Y](http://www.youtube.com/watch?v=F8Do9bDfd1Y)).

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4Educologists will surely agree with the Greeks that, when all is said and done, the only real numbers are the counting numbers.

5Still, the difference can be difficult to pin down. As Educologists well know, while the ancient Greeks had the concept of number, they were unable to deal with amounts other than via ‘ratios and proportions’ which effectively stopped the development of Arithmetic for many centuries. Which in turn makes one wonder why Educologists should keep insisting on teaching “ratios and proportions”.

<table>
<thead>
<tr>
<th>entities</th>
<th>quantities</th>
</tr>
</thead>
<tbody>
<tr>
<td>items</td>
<td>numbers</td>
</tr>
<tr>
<td>substances</td>
<td>amounts</td>
</tr>
</tbody>
</table>
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