What is important is the real-world, that is physics, but it can be explained only in mathematical terms.

Dennis Serre[1]

margin
index
real-world
entity
relationship
information
action

Chapter 1

Symbolic Systems

LANGUAGE 1.1. Just as in any **S**cience, **T**echnology and **E**ngineering, the answer in **M**athematics to the question "You know what I mean?" is always "No!" because we need to agree, precisely and exactly, on *what* we are talking about. In particular, the reader will have to know the meaning of the words that are boldfaced as they are introduced in this text. To help the reader who, later on, might not be too sure about such a word, the page where it was introduced can always be found in the **index** at the end of the boook and the word will also appear at the top of the **margin** on the page where it was first introduced.

Contrary to what most people think, MATHEMATICS in general, and ARITHMETIC in particular, originate in the **real-world** and deal with real-world **entities** and the **relationships** among real-world entities. However, in order to:

- Communicate about the real-world, that is share **information** about the real world,
- Investigate how the real-world *works*,
- Plan in advance our **actions** on the real-world (because acting on the real-world without thinking ahead usually has very bad consequences),

[1]Bulletin of the American Mathematical Society, Vol 47 Number 1 Pages 139-144

language
represent
word
vocabulary
noun
verb
phrase
noun phrase
verb phrase
hyphen

we need a **language** to **represent** the real world on *paper*.

And, indeed, there are all sorts of languages: anywhere between 3000 and 8000 spoken languages, hundreds of sign-languages, etc. There are written languages, pictorial languages, secret and non-secret codes, bar codes, computer languages, etc. (See http://en.wikipedia.org/wiki/Language.)

Nevertheless, as we will see, MATHEMATICS requires a language of its own.

1.1 What Languages Are Made Of

Very, very roughly, most languages are constructed as follows:

1. The building blocks of the language are **words** listed in a **vocabulary** that must be available to all users of the language.

EXAMPLE 1.1.1. Of the following:

- word, wood
- woord, wod

the first two are English words but the other two are not.

The two main kinds of *words* that we will be using are:

– **Nouns** to represent real-world *entities*,
– **Verbs** to represent *relationships* among real-word entities.

EXAMPLE 1.1.2. Of the following:

- apple, triangle
- eats, intersects

the first two are *nouns* and the other two are *verbs*.

2. Quite often, though, we do not have single ready-made words to represent real-word entities or to represent a relationship among real-word entities and we will have to use a **phrase**, that is "a small group of words standing together as a conceptual unit" (http://www.oxforddictionaries.com/us/definition/american_english/phrase). More precisely, we will use:

– **Noun phrases**, that is groups of words that work the same way single nouns do, to represent real-world entities,
– **Verb phrases**, that is groups of words that work the same way single verbs do, to represent relationships among real-word entities.

We will place **hyphens** between the words that make up a phrase to emphasize the fact that these words are to be read together as a whole.

EXAMPLE 1.1.3. Of the following:

- ten-dollar-bill, mathematics-teacher
- takes-apart, is-the-daughter-of

the first two are *noun phrases* and the other two are *verb phrases*.

3. In order to communicate we use **sentences**, that is groups of *words* or *phrases* constructed according to **grammatical rules** listed in a **grammar** that must be available to all users of the language.

sentence
grammatical rule
grammar
state
statement
meaning
meaningless
TRUE
occur
FALSE
procedure
simulate
process

EXAMPLE 1.1.4. Of the following:

- The hamburger paints the ocean with the mountains.
- Dog fox lazy quick brown the jumps the over.

the first is an English *sentence* but the second is not an English sentence.

4. However, a *sentence* may or may not be **stating** something about the real-world. A *sentence* that states something about the real-world is called a **statement** and what it states about the real-world is called the **meaning** of the sentence. A sentence that does not state something about the real-world is said to be **meaningless**.

EXAMPLE 1.1.5. Of the following two *sentences*:

- The spaghetti sauce is too hot to be eaten with a spoon.
- The rose understands the navy with a table.

the first is a *statement* (about the real-world) but the second is *meaningless* since it is *not* a statement (about the real-world).

5. A sentence is then:

– **TRUE** if what it states about the real-world **occurs** in the real-world,
– **FALSE** if what it states about the real-world does *not* occur in the real-world.

EXAMPLE 1.1.6. The following are both (famous) *statements* about the real-world:

- The moon is made of green cheese.
- Humans are the only featherless bipeds.

However,

> The first statement is FALSE because, as Armstrong checked on July 20, 1969, the moon is *not* made of green cheese.

while

> The second statement is TRUE because, even though there are many species that are featherless and many species that use only two legs for walking, it so happens that humans are the only ones to be both featherless and bipedal.

6. Our main goal in this text will be, using *sentences*, to develop **procedures** to **simulate** on paper real-world **processes**.

EXAMPLE 1.1.7. Suppose we have to carry out the real-world *process* of handing out seventeen dollars to each one of two thousand six hundred and forty eight people. A preliminary issue then is whether we have the cash to carry out this real-world process. To find out, we *represent* on paper both how many dollars per person and how many persons and we then use the paper-world *procedure* called *multiplication* to figure out on paper how many dollars we will need in order to carry out the *process* in the real world.

native language
object language
metalanguage
pictures
substitute

1.2 Learning Languages

Learning a language is nowhere as simple as one might think.

1. The language that each one of us first learned as a child, our **native language**, we learned by directly associating *nouns* with the real-world *entities* that the nouns represent. But how children learn *verbs* to represent real-world *relationships* among real-world entities still isn't fully understood.

2. A second language is usually not learned the same way as a native language, that is not by direct association with the real-world, but through the language that we already know. So, learning a second language automatically involves *two* languages which we will need to keep separate:

– The **object language**, which is the language that we want to learn,
– The **metalanguage**, which is the language we already know and which we use to learn the object language.

Example 1.1.8. For an American learning Spanish, the *object language* is Spanish and English is the *metalanguage* but for a Spaniard learning English, the *object language* is English and Spanish is the *metalanguage*.

3. In a book, the *metalanguage* is used for several different purposes:

- To describe and discuss the *real-world* to be to be represented on paper with the object language. In particular, we will use the *metalanguage* to describe and discuss:
  - The real-world *entities* to be represented on paper,
  - The real-world *relationships* among real-world entities to be represented on paper;

but also

- To describe and discuss features of the *object language* itself,

and, more generally,

- To discuss matters.

1.3 In This Book

We will be using English as our metalanguage.

1. Since we will not be able to exhibit the real-world items that we will want to represent on paper, we will usually use **pictures** as **substitutes** for the real-world entities. However, as even using pictures will not always be possible, we will also use as substitutes English words which will then be printed in ***bold-faced italics***. So, in this book, words printed in bold-faced italics will belong to the metalanguage.

shorthand
boldfaced
margin
index
natural language

EXAMPLE 1.1.9. Say we intend to discuss the paper-representation of *real-world* one-dollar bills. Since we cannot exhibit these *real-world* one-dollar bills, we will use *pictures* as a *substitute* for these real-world one-dollar bills, for instance

However we will also use as *substitutes* for these real-world one-dollar bills the English words (in bold-faced italics),

one-dollar bill ***one-dollar bill*** ***one-dollar bill***

NOTE. It is very important to distinguish *nouns* from *substitutes*:

- *nouns* are intended to *represent on paper* the real-world items.

while

- *substitutes* are intended to *stand for* the real-world items and will be used only when we cannot use *pictures* to do so.

2. When there will be too many real-world entities for us to use pictures or English words as substitutes, we will use English numbers in SMALL CAPS as a **shorthand**. So, in this book, numbers spelled out in small caps belong to the metalanguage.

EXAMPLE 1.1.10. We will use FIVE ***apples*** as a shorthand for ***apple apple apple apple apple*** as well as FIVE  as a shorthand for .

3. In order to make it easier to find in this book the place where words, both in the object language and in the metalanguage, appear for the first time and are explained, this book uses a standard method: These words are

– **boldfaced** the first time they appear in this book and are explained,
– printed in the **margin** of the page where they appear for the first time and are explained,
– listed in the **index** at the end of the book along with the number of the page where they appear for the first time and are explained.

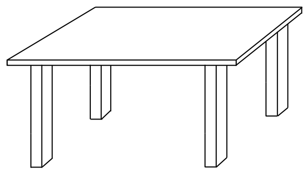
1.4 Natural Languages

A **natural language** is the kind of language we use most of the time, routinely, and rather loosely. Natural languages depend on place and time.

EXAMPLE 1.1.11.

specialized languages

The real-world entity whose *picture* is



is represented by different *words* depending on the natural language we are using:

Language	Word
Chinese	表
English	table
Finnish	taulukko
French	table
German	Tisch
Greek	τραπέζι
Italian	tavola
Japanese	食卓
Latin	tabula
Russian	стол
Spanish	mesa
Vietnamese	bảng

How natural languages were born and how they then evolved and interacted with one another is buried in time and investigated in ETYMOLOGY (See http://en.wikipedia.org/wiki/Etymology).

1.5 Specialized Languages

While natural languages usually serve us tolerably well for general communication, they have severe limitations.

1. People working in any trade need **specialized languages** because:

– Workers need to know exactly what they are to do,

EXAMPLE 1.1.12. We cannot just tell an architect “Design me a building”. Architects need *specifications*. In turn, architects cannot just give pictures to construction workers. Construction workers need at least *blueprints*.

– Workers need to communicate with each other while doing the work.

EXAMPLE 1.1.13. Imagine the likely consequences of two persons working with some dangerous machinery and one telling the other: “Push the gizmo which is on the whatchamacallit next to the doodad on the other side of the doohickey.”

2. Sometimes, specialized languages use made-up words but most of the time they use common words with a meaning special to the trade.

EXAMPLE 1.1.14. When an *electrician* asks for a pancake, that's not because s/he is hungry but because s/he needs a junction box the thickness of drywall. When a *carpenter* asks for *drywall*, s/he is not likely to be asking for a wall that is not wet. When a *mason* asks for a *hawk*, s/he is not likely to be asking for a bird.

EXAMPLE 1.1.15. “[The actor] must learn the theatre's special vocabulary. Partly technical, partly slang, much of it is standardized on the English-speaking stage. As a working actor, you must be familiar with this language, just as a mechanic must know the names of his tools or a surgeon the names of her instruments.” (From *Acting is Believing* by Charles McGaw, Kenneth L. Stilson, Larry D. Clark.)

EXAMPLE 1.1.16. "If you look up the meaning of the word "theory" in the dictionary, it is described as being a synonym for words like "proposition", "hypothesis", or even "speculation". In contrast, a scientific theory is an established body of knowledge about a certain subject, supported by observable facts, repeatable experiments, and logical reasoning. A theory in science is a formal explanation of some aspect of the natural world, tested and verified by careful observation and experimentation. A good theory is one that also produces accurate and useful predictions." (From https://plus.maths.org/content/evolution-its-real-gravity?nl=0.)

make-the-case
correct
compute
deductive rules
systematic
symbol
symbolic-system
specified
noun-symbol
verb-symbol
&
$\vee$
$\neg$
symbolic-sentence

1.6 Mathematical Language

In the mathematical trade, the situation is exactly the same as in any other trade except even more so because, in mathematics, just as if we were in a court of law, we *always* have to **make-the-case** that:

– The *mathematical sentences* we write are TRUE, that is are statements about the real-world whose meaning occurs in the real-world,
– The *mathematical procedures* we develop are **correct**, that is simulate the real-world processes.

1. As a result, we will have to develop an object language that makes it as easy as possible for us to **compute**, that is to manipulate the words in the object language according to listed **deductive rules**. So, the main feature we will require from the object language is that it be **systematic**.

2. In order to stress the *systematic* aspect of the object language that we will develop, we will often use **symbols** instead of *words* and we will use the phrase **symbolic-system** instead of the word *language*. More precisely, a symbolic-system is **specified** by a *vocabulary* of symbols:

– **Noun-symbols** to represent real-world *entities*,
– **Verb-symbols** to represent *relationships* among real-world entities,

together with

– **Logical-symbols** to represent in the symbolic-system the meaning of the English words "and", "or", "not".[2]. We will use &, $\vee$, $\neg$

along with a *grammar*.

3. We can then assemble these symbols to write **symbolic-sentences**. These symbolic sentences will then be TRUE or FALSE depending on whether they make statements about the real-world whose meaning occurs or does

[2] Educologists will surely object to our excluding "if ... then ...". The reason is that, instead of using implication (a binary operation among sentences), we find it more natural to say that a sentence *yields* another sentence (a binary relation among sentences) to represent on paper *logical consequence*.

code
comment

not occur in the real-world. We will then say that we have **coded** the information about the real-world into the *symbolic-system*.

EXAMPLE 1.1.17. Let's say we want to *code* the following real-world situation:

- The entities are three persons: ***Andy***, ***Beth*** and ***Cathy***.
- The relationship is ***loves/hates*** as determined by the following table:

loves/hates $\uparrow$	***Andy***	***Beth***	***Cathy***
Andy	***loves***	***loves***	***loves***
Beth	***loves***	***hates***	***loves***
Cathy	***hates***	***loves***	***hates***

So, for instance, in the real-world according to the above table, we have that ***Andy loves Cathy***, ***Cathy hates Andy***, etc.

i. In order to *code* the information about this real-world situation, we may, for instance, use the following symbolic-system:

- The *vocabulary* consists of the following symbols:
  - Noun-symbols: a to represent ***Andy***, b to represent ***Beth***, c to represent ***Cathy***.
  - Verb-symbol: L to represent ***loves***,
  - Logical-symbol: & to represent ***and***.
- The *grammar* says that the noun-symbols should be on either sides of the verb-symbol—just as in English. (But, for another possible *grammar*, see http://en.wikipedia.org/wiki/Reverse_Polish_notation.)

ii. Then, for instance:

- The symbolic-sentence

$$bLa$$

  means ***Beth loves Andy*** and since, in the real-world according to the above table, ***Beth loves Andy***, the symbolic-sentence bLa is TRUE.
- The symbolic-sentence

$$cLa$$

  means ***Cathy loves Andy*** and since, in the real-world according to the above table, ***Cathy hates Andy***, the symbolic-sentence cLa is FALSE.
- The symbolic-sentence

$$cLb \,\&\, cLa$$

  means that ***Cathy loves Beth and Cathy loves Andy*** and since, in the real-world according to the above table, ***Cathy loves Beth*** but ***Cathy hates Andy***, the symbolic-sentence $cLb \,\&\, cLa$ is FALSE.

NOTE. In *this* symbolic-system, we would not be able to write a symbolic-sentence to mean that ***Cathy loves Beth or Cathy loves Andy*** because the vocabulary in *this* symbolic-system does not include a logical-symbol to represent ***or***.

4. Aside from the need to compute with symbols, we will also want to be able to make **comments** in the metalanguage in order to describe, explain and discuss what it is that we are doing within the *symbolic-system*.

EXAMPLE 1.1.18. Software engineers will tell you that any computer code *must* include comments about what the code is supposed to do that are intended to be read by *humans*—while being ignored by the *computer*. Indeed, in the absence of comments, the code rapidly becomes unusable because after a while the way it works cannot be understood anymore and therefore the code cannot be maintained. So, every computer language includes a way for there to be *comments*.

declared quantity

5. Finally, we have to agree to write in the *standard* mathematical language. This is because we are trying to write sentences that are TRUE and the only way to make sure that our sentences are TRUE is to submit to others the *case* we made for their truth. And of course, if we do not write in the standard mathematical language, nobody is going to look at, even less debate, our case and so we will never know if the sentences we wrote are TRUE or FALSE. See `http://en.wikipedia.org/wiki/Peer_review`

However, mathematicians do allow deviations from and additions to the standard mathematical language because:

i. A symbolic-system for dealing with all of mathematics would be so big as to be totally unusable,

and so,

ii. Given a particular purpose, parts of the *standard* mathematical language have to be re-cycled and, more generally, deviations are often unavoidable.

Of course, as in computer programming, any deviation from and/or addition to the standard mathematical language has to be **declared** *up front*. And a look at any *real* mathematical text will show how often mathematicians use the word *definition*.

1.7 Symbolic Systems for Arithmetic

As opposed to GEOMETRY which deals with the *shape* of entities, KINEMATICS which deals with the *speed* of entities, DYNAMICS which deals with the way entities react to *forces*, THERMODYNAMICS which deals with the temperature of entities, etc, ARITHMETIC deals with **quantities** of entities.

1. What immediately complicates matters, though, is that, there are two very different kinds of real world *entities* and, correspondingly, two very different kinds of *quantities* [3]:

[3] Here, Educologists would of course speak of *multitude* versus *magnitude*. The trouble, though, is that both terms usually connote "big".

item
number
substance
amount

entities	*quantities*
items	**numbers**
substances	**amounts**

- *Items* are distinct entities we can deal with *individually* and instead of speaking of a quantity of *items* we speak of a *number*[4] of *items*.
  Example 1.1.19. ***Apples*** are *items* we can deal with one at a time and we speak of a *number* of ***apples***.
- A *substance* is an entity we can deal with only *in bulk* and instead of speaking of a quantity of *substance* we speak of an *amount* of *substance*.
  Example 1.1.20. ***Soup*** is a *substance* we can deal with only in bulk and we speak of an *amount* of ***soup***.

2. Natural languages often use different words when dealing with *numbers* and when dealing with *amounts* but not always.

Example 1.1.21. In English, the distinction between *numbers* and *amounts* is not always made:

	Numbers	Amounts
	Too *many* ***apples***	Too *much* ***soup***
	A *few* ***apples***	A *little* ***soup***
	Fewer ***apples*** than ***bananas***	*Less* ***soup*** than ***milk***
but:	*More* ***apples*** than ***bananas***	*More* ***soup*** than ***milk***
in spite of:	*Many more* ***apples*** than ***bananas***	*Much more* ***soup*** than ***milk***

3. As we will see, the distinction between *numbers* and *amounts* is an absolutely fundamental one, especially in sciences and technologies[5]. (See http://www.differencebetween.net/science/mathematics-statistics/difference-between-number-and-amount/ and http://www.youtube.com/watch?v=F8Do9bDfd1Y).

[4]Educologists will surely agree with the Greeks that, when all is said and done, the only *real* numbers are the *counting* numbers.

[5]Still, the difference can be difficult to pin down. As Educologists well know, while the ancient Greeks had the concept of *number*, they were unable to deal with *amounts* other than via 'ratios and proportions' which effectively stopped the development of Arithmetic for many centuries. Which in turn makes one wonder why Educologists should keep insisting on teaching "ratios and proportions".

Index