Chapter 10

Multiplication - Division of Signed Numbers

Just like multiplication and division of plain numbers were very different from addition and subtraction of plain numbers, multiplication and division of signed numerators are very different from addition and subtraction of signed numbers.

i. Multiplication of signed numbers cannot be repeated *oplussing*.

Example 10.1. What repeated *oplussing* could -3 **Arrows** multiplied by $+5$ **Arrows** possibly stand for?

ii. Multiplication of signed numbers cannot give an area.

Example 10.2. What area could -3 **Feet** multiplied by $+5$ **Feet** possibly be? The denominator would of course have to be **SquareFeet** but what if the result of the multiplication turned out *negative*? What could a *negative* area be?

10.1 Signed Co-multiplication

This is where multiplication of signed numerators, for which we will use the symbol $\otimes$, is very useful.

signed co-multiplication

1. We begin by looking at the real-world. As before, we want to investigate the *change* in a given state, *gain* or *loss*, that results from a given transaction, "in" or "out" as before but with *oriented collections* of "good" items or "bad" items.

EXAMPLE 10.3. Consider a store where collections of ***apples*** can either ***get* into *the store*** or ***get* out of *the store***. Moreover, the collections are collections of *oriented* items in that the apples can be either ***good apples***—inasmuch as they will generate a *profit* when they are sold—or ***bad apples***—inasmuch as they will generate a *loss* because they will have to be disposed of at a cost.

2. We now look at the way we will represent things on paper.

a. We will represent

- Collections of items getting "in" by *positive* number-phrases,
- Collections of items getting "out" by *negative* number-phrases,

EXAMPLE 10.4. In EXAMPLE 10.3 , we would represent

- Collections of ***apples*** getting *into* the store by *positive* number-phrases,
- Collections of ***apples*** getting *out of* the store by *negative* number-phrases,

b. We will represent

- The unit-worth of "good" items by a *positive* number-phrase,
- The unit-worth of "bad" items by *negative* number-phrases,

EXAMPLE 10.5. In EXAMPLE 10.3 , we would represent

- The ***unit-value*** of ***good apples*** that is ***apples*** that will generate a sales *profit* of seven cents per apples by the co-number-phrase $+7\ \frac{\text{Cents}}{\text{Apple}}$
- The ***unit-value*** of ***bad apples*** that is ***apples*** that will generate a disposal *cost* of seven cents per apple by the co-number-phrase $-7\ \frac{\text{Cents}}{\text{Apple}}$

3. We can now write the procedure for **signed co-multiplication** for which we will use the symbol $\otimes$:

i. multiply the *denominators* (with cancellation).

ii. *Otime* the *numerators* according to the way the result is a "good" change or a "bad" change:

- A collection of "good" items getting "in" makes for a "good" change so $+ \otimes + = +$.

Example 10.6.

Three apples get *in* the store. $+3$ **Apples**

The apples have a unit-value of seven cents-per-apple *gain*. $+7\ \frac{\textbf{Cents}}{\textbf{Apple}}$

The specifying phrase is $[+3\ \textbf{Apples}] \otimes \left[+7\ \frac{\textbf{Cents}}{\textbf{Apple}}\right]$

We co-multiply $[(+3) \otimes (+7)]\ \left[\cancel{\textbf{Apples}} \times \frac{\textbf{Cents}}{\cancel{\textbf{Apple}}}\right]$

We get a twenty-one cent *gain*. $= +21$ **Cents**

- A collection of “good” items getting “out” makes for a “bad” change so $+ \otimes - = -$.

Example 10.7.

Three apples get *in* the store. $+3$ **Apples**

The apples have a unit-value of seven cents-per-apple *loss*. $-7\ \frac{\textbf{Cents}}{\textbf{Apple}}$

The specifying phrase is $[+3\ \textbf{Apples}] \otimes \left[-7\ \frac{\textbf{Cents}}{\textbf{Apple}}\right]$

We co-multiply $[(+3) \otimes (-7)]\ \left[\cancel{\textbf{Apples}} \times \frac{\textbf{Cents}}{\cancel{\textbf{Apple}}}\right]$

We get a twenty-one cent *loss*. $= -21$ **Cents**

- A collection of “bad” items getting “in” makes for a “bad” change so $- \otimes + = -$.

Example 10.8.

Three apples get *out* of the store. -3 **Apples**

The apples have a unit-value of seven cents-per-apple *gain*. $+7\ \frac{\textbf{Cents}}{\textbf{Apple}}$

The specifying phrase is $[-3\ \textbf{Apples}] \otimes \left[+7\ \frac{\textbf{Cents}}{\textbf{Apple}}\right]$

We co-multiply $[(-3) \otimes (+7)]\ \left[\cancel{\textbf{Apples}} \times \frac{\textbf{Cents}}{\cancel{\textbf{Apple}}}\right]$

We get a twenty-one cent *loss*. $= -21$ **Cents**

- A collection of “bad” items getting “out” makes for a “good” change so $- \otimes - = +$.

Example 10.9.

Three apples get *out* of the store.	-3 **Apples**
The apples have a unit-value of seven cents-per-apple *loss*.	$-7\,\frac{\textbf{Cents}}{\textbf{Apple}}$
The specifying phrase is	$[-3\ \textbf{Apples}] \otimes \left[-7\,\frac{\textbf{Cents}}{\textbf{Apple}}\right]$
We co-multiply	$[(-3) \otimes (-7)]\ \left[\cancel{\textbf{Apples}} \times \frac{\textbf{Cents}}{\cancel{\textbf{Apple}}}\right]$
We get a twenty-one cent *gain*.	$= +21$ **Cents**

In other words, the rule for the *multiplication of signs* is:

THEOREM 10.1 Multiplication of Signs

	+	−
+	+	−
−	−	+

10.2 Signed Division

The rule of signs for the *division of signs* is the same as the rule for the *multiplication of signs*:

THEOREM 10.2 Division of Signs

	+	−
+	+	−
−	−	+

EXAMPLE 10.10. If we know that the store has incurred a TWENTY-ONE ***cents*** loss with THREE ***apples*** moving out, what was the unit-worth of the ***apples***?
Dividing -21 **Cents** by -3 **Apples** gives us $+7\frac{\textbf{Cents}}{\textbf{Apple}}$
(Indeed, if ***apples*** moving *out* resulted in a *loss*, then the ***apples*** must have been *good*!)

EXAMPLE 10.11. If we know that the store has incurred a TWENTY-ONE ***cents*** loss with ***good apples*** worth SEVEN ***Cents*** each, how many ***apples*** moved and which way did they move?

Dividing -21 **Cents** by $+7\frac{\textsf{Cents}}{\textsf{Apple}}$ gives us -3 **Apples**
(Indeed, if ***good apples*** moving resulted in a *loss*, then the ***apples*** must have moved *out*!)

EXAMPLE 10.12. If we know that the store has incurred a TWENTY-ONE ***cents*** gain with THREE ***apples*** moving out, what is the unit-worth of the ***apples***?
Dividing $+21$ **Cents** by -3 **Apples** gives us $-7\frac{\textsf{Cents}}{\textsf{Apple}}$
(Indeed, if ***apples*** moving *out* resulted in a *gain*, then the ***apples*** must have been *bad*!)

EXAMPLE 10.13. If we know that the store has incurred a TWENTY-ONE ***cents*** gain with the unit-worth of the ***bad apples*** worth SEVEN ***Cents*** each, how many ***apples*** moved and which way did they move?
Dividing $+21$ **Cents** by $-7\frac{\textsf{Cents}}{\textsf{Apple}}$ gives us -3 **Apples**
(Indeed, if ***bad apples*** moving resulted in a *gain*, then the ***apples*** must have moved *out*!)

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