Chapter 12

Fractions – Comparison

There are two main aspects to fractions

- Fractions can be a historical remnant from before the invention of the decimal-metric system
- Fractions can be *code* for division.

but unfortunately fractions are also confused very often with

- scores.

12.1 Fractions as Historical Leftovers

1. Imagine someone who knows only

- *counting numerators* namely $1, 2, 3, \ldots$
- and that the *denominator* **Dollar** represents a real world 

Suppose now that this person were to ask what  is.
The usual responses are, more or less:

- "A **Quarter**". This, though, just says what the *denominator* for this item is. It does not say *what* the real world item itself *is*.
- "25 **Cents**". Since the person knows how to count, s/he understands the *numerator* 25 but not the *denominator* **Cents**. So, this does not say *what*

the real world item itself *is*.

- "$\frac{1}{4}$ **Dollar**". Since the person knows what a **Dollar** is s/he understands the *denominator* but since s/he only knows how to *count* s/he doesn't understand what $\frac{1}{4}$ means. So, this does not say *what* the real world item itself *is*.

The only things that will explain to this person what a quarter is is to say that four such items can be exchanged for a dollar:

$=$

In other words, we can only write:

$$4 \textbf{ Quarters} = 1 \textbf{ Dollar}$$

2. The difficulty though is that this does not allow us to use "quarter" as a *denominator*.

Example 12.1. After we have explained what a ***Lincoln*** is in terms of what a ***Washington*** is by writing:

$$1 \textbf{ Lincoln} = 5 \textbf{ Washingtons}$$

we can explain what a number of ***Lincolns*** is by writing:

$$2 \textbf{ Lincoln} = 10 \textbf{ Washingtons}$$
$$3 \textbf{ Lincoln} = 15 \textbf{ Washingtons}$$
$$4 \textbf{ Lincoln} = 20 \textbf{ Washingtons}$$
$$5 \textbf{ Lincoln} = 25 \textbf{ Washingtons}$$

etc

Example 12.2. But after we have explained what a ***Quarter*** is in terms of what a ***Washington*** is by writing:

$$4 \textbf{ Quarters} = 1 \textbf{ Dollar}$$

we still cannot explain what most numbers of ***Quarters*** are:

$$2\ \textbf{Quarters} = ?$$
$$3\ \textbf{Quarters} = ?$$
$$4\ \textbf{Quarters} = 1\ \textbf{Dollar}$$
$$5\ \textbf{Quarters} = ?$$
etc

because the 1 in what we wrote to explain what a ***Quarter*** is in terms of what a ***Washington*** is on the wrong side.

3. The only way out is to use a standard *linguistic trick*.

Example 12.3. Use **Quarter** as a shorthand for:

of-which-4-can-be-exchanged-for-1-Dollar

Then, we can explain what all numbers of ***Quarters*** are:

2 **Quarters** = 2 **of-which-4-can-be-exchanged-for-1-Dollar**

3 **Quarters** = 3 **of-which-4-can-be-exchanged-for-1-Dollar**

4 **Quarters** = 1 **Dollar**

5 **Quarters** = 5 **of-which-4-can-be-exchanged-for-1-Dollar**

etc

4. An immediate advantage is that it also makes it will make it easier to compare *fractions* with *counting numerators*.

Example 12.4. We can see that

5 **of-which-4-can-be-exchanged-for-1-Dollar**

is the same amount of money as

1 **Dollar** & 1 **of-which-4-can-be-exchanged-for-1-Dollar**

fraction bar

and then we have:

2 **Quarters** = 2 **of-which-4-can-be-exchanged-for-1-Dollar**

3 **Quarters** = 3 **of-which-4-can-be-exchanged-for-1-Dollar**

4 **Quarters** = 1 **Dollar**

5 **Quarters** = 1 **Dollar** & 1 **of-which-4-can-be-exchanged-for-1-Dollar**

6 **Quarters** = 1 **Dollar** & 2 **of-which-4-can-be-exchanged-for-1-Dollar**

7 **Quarters** = 1 **Dollar** & 3 **of-which-4-can-be-exchanged-for-1-Dollar**

8 **Quarters** = 2 **Dollars**

9 **Quarters** = 2 **Dollar** & 1 **of-which-4-can-be-exchanged-for-1-Dollar**

etc

5. However, the above is of course not the way we usually write fractions because it would make it awkward to develop procedures for calculating with fractions and we are now going to see how we came about to write, and what is involved when we *write*, say,

$$\frac{3}{4}\ \textbf{Dollar}$$

Starting from

3 **Quarters**

we already saw above that this is

3 **of-which-4-can-be-exchanged for-1-Dollar**

Now we write the denominator **of-which-4-can-be-exchanged for-1-Dollar** symbolically as $\boxed{4 \rightarrow 1\ \textbf{Dollar}}$ so that we now have

$$3\ \boxed{4 \rightarrow 1\ \textbf{Dollar}}$$

What happened at this point is that, instead of writing the *denominator* next to the *numerator*, the *denominator* came to be written under the *numerator*:

$$\frac{3}{\boxed{4 \rightarrow 1\ \textbf{Dollar}}}$$

and then of course the box disappeared since the **fraction bar** was enough of a separator:

$$\frac{3}{4 \rightarrow 1\ \textbf{Dollar}}$$

But then, part of the denominator went back to the right of the numerator which required the arrow to be bent

$\frac{3}{4}$ 1 **Dollar**

and, as usual, the 1 then went without saying

$\frac{3}{4}$ **Dollar**

and finally the arrow disappeared which gives us what we usually write

$$\frac{3}{4}\ \textbf{Dollar}$$

If we look at this as "3", "bar", "4", "**Dollars**" in that order, then the fraction bar continues to act as a separator between "3", the *numerator*, and "4", "**Dollars**", the *denominator* which, to make sense, must still be read as "4 for a **Dollars**", in other words, we have

3 **Items-at-**4**-for-a-Dollar**

Thus, while 3 is truly a *numerator*, 4 is only *part* of the *denominator* and, if we forget this, then it becomes impossible to deal with fractions on the basis of common sense.

Altogether, it is crucial to see

$$\frac{3}{4}\ \textbf{Dollar}$$

as just a *shorthand* for

3 **Items-at-**4**-for-a-Dollar**

12.2 Mixed-Numbers and Improper Fractions

improper fraction

For some reason, by now lost in time, number-phrases such as

4 **Quarters**

5 **Quarters**

6 **Quarters**

etc

are called in school language **improper fractions** even though there is nothing wrong with them.

1. Converting "improper fractions", for instance $\frac{9}{4}$ **Dollar**, to something presumably more "proper" is a favorite school exercise but is absolutely straightforward if we keep in mind that

- the *numerator* is 9
- the *denominator* is **Of-which-**4**-can-be-exchanged-for-**1**-Dollar**

mixed number

Then,

$$\frac{9}{4}\ \textsf{Dollar} = 9\ \textsf{Of-which-4-can-be-exchanged for-1-Dollar}$$
$$= 2\ \textsf{Dollars}\ \&\ 1\ \textsf{Of-which-4-can-be-exchanged for-1-Dollar}$$

which can be coded as

$$= 2\ \textsf{Dollar}\ \&\ \frac{1}{4}\ \textsf{Dollar}$$

so that, since the denominators are the same,

$$= \left[2 + \frac{1}{4}\right]\ \textsf{Dollars}$$

which in fact is often written

$$= 2\tfrac{1}{4}\ \textsf{Dollars}$$

where $2\frac{1}{4}$ is called a **mixed number** and where the fact that the fraction is written with smaller digits is supposed to warn that the missing operation sign is a $+$.

2. Converting “mixed-numbers” to “improper fractions” is an equally popular exercise in school and just as straightforward as the former one. For instance,

$$2\tfrac{1}{4}\ \textsf{Dollars} = \left[2 + \frac{1}{4}\right]\ \textsf{Dollars}$$
$$= 2\ \textsf{Dollar}\ \&\ \frac{1}{4}\ \textsf{Dollar}$$
$$= 2\ \textsf{Dollars}\ \&\ 1\ \textsf{Of-which-4-can-be-exchanged for-1-Dollar}$$

which tells us at what rate to change the 2 **Dollars**

$$= 8\ \textsf{Of-which-4-can-be-exchanged for-1-Dollar}\ \&\ 1\ \textsf{Of-which-4-can-be-exchanged for-1-Dollar}$$

which we can code as

$$= \frac{8}{4}\ \textsf{Dollar} + \frac{1}{4}\ \textsf{Dollar}$$

and, since the denominators are the same,

$$= \frac{9}{4}\ \textsf{Dollar}$$

12.3 Fractions as Code for Division

These days, other than in the schools and the use of “half”, “quarter”[1], “eighth” etc in construction, fractions are now mostly used as code for *divi-*

[1]Notice by the way the demise of “half”, “quarter” brought about by digital watches: who still says “a quarter to two”?

sion.

12.4 Comparison of Fractions

The safest and usually fastest way to *compare* fractions is to carry out both divisions to enough digits that the quotients become different.

EXAMPLE 12.5. To *compare* $\frac{5}{7}$ and $\frac{17}{25}$, do both divisions until the quotients become differents:

i. Doing the divisions to the "ones":

- $5 \div 7 = 0. + [...]$
- $17 \div 25 = 0. + [...]$

Therefore continue the divisions

ii. Doing the divisions to the "tenths":

- $5 \div 7 = 0.7 + [...]$
- $17 \div 25 = 0.6 + [...]$

Therefore $\frac{5}{7} > \frac{17}{25}$

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