There are only four systems that bridge all linguistic barriers:
–The complete set of mathematical symbols,
–The International System of Units,
–The symbols for chemical elements,
–The way of writing notes for music.

Anders J. Thor[1]

lump
length
area
volume

Chapter 3

Decimals for Amounts of Substance

While *items* come in as *collections*, *substances* come in as **lumps** but natural languages use specialized words instead of the generic word "lump". For instance, in English:

- A 1-dimensional *lump* is usually called a **length**. (See http://en.wikipedia.org/wiki/Length.)
- A 2-dimensional *lump* is usually called an **area**. (See http://en.wikipedia.org/wiki/Area.)
- A 3-dimensional *lump* is usually called a **volume**. (See http://en.wikipedia.org/wiki/Volume.)

Example 3.3.1.

- A *length* of rope.
- An *area* of grass.
- A *volume* of water.

In order to *represent* a *lump* on paper, we will again write a *number phrase* that gives all the *information* necessary to specify that *lump*:

- The *kind* of substance in the lump (*qualitative* information) which we represent on paper by a *denominator* that is a name usually associated with the substance,
- The *amount* of substance in the lump (*quantitative* information) which we

[0]Chairman of IEC TC 25

reduce
measure
discretize
unit lump
measured part
measure

will represent by a *numerator*. In the case of *lumps*, though, getting the numerator will be a bit more complicated than in the case of *collections* where we just had to *count* the items.

3.1 Measuring Amounts

To deal with the problem of how to represent *amounts of substances* we will use a standard approach in mathematics which is to **reduce** a new problem to a problem we already solved.

1. In the present case, we will reduce the problem of representing an *amount of substance* to the problem of representing a *number of items*, which we already solved in **??**. We will do that by first **measuring** the *lump of substance* which we do as follows:

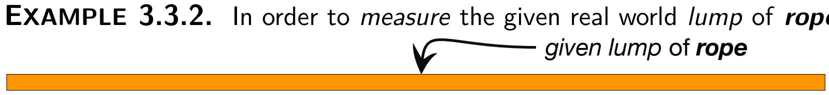
i. We **discretize** the lump of substance, that is:

– We choose a **unit lump** of substance.

– We determine the **measured part** of the given lump, that is the largest collection of *unit lumps* contained in the given *lump*, where by largest we mean that, with one more *unit lump*, we would have more substance than in the given *lump*.

ii. The **measure** of the given lump is then the number of *unit lumps* in the *measured part*.

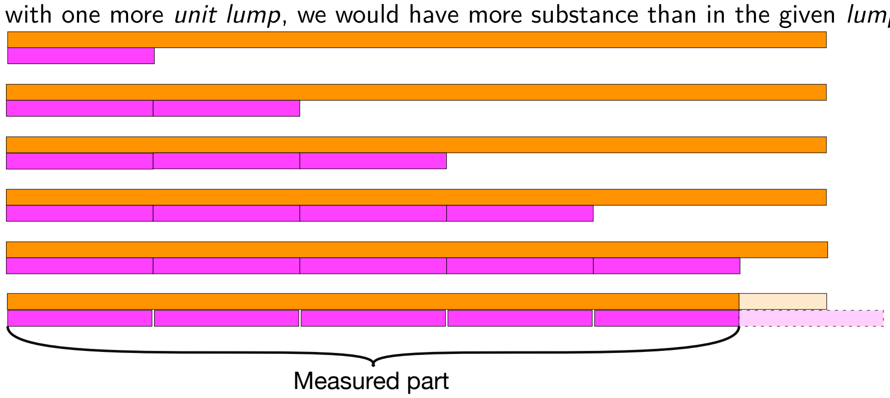
EXAMPLE 3.3.2. In order to *measure* the given real world *lump* of ***rope***,



i. We discretize the given real world *lump* of ***rope***, that is:

– We choose a *unit lump* of rope, for instance 

– We determine the *measured part* of the given lump which is the largest collection of *unit lumps* contained in the given *lump*, where by largest we mean that, with one more *unit lump*, we would have more substance than in the given *lump*:



ii. The *measure* of the given lump of ***rope*** is therefore FIVE *unit lumps* of ***rope***.

remainder [...]

2. Representing lumps involving a *large* amount of real world substance is not a problem: since the *measured part* of a lump is a *collection* of *unit lumps*, the procedure for denoting a *measured part* with a *large number of unit lumps* will be exactly the same as the procedure we used in **??** to represent a collection of a *large number of items*.

EXAMPLE 3.3.3. The *measured part* of a large length of ***rope*** might be represented on paper by a *number phrase* like:

[7 THOUSAND, 5 HUNDRED, 8 TEN, 3 SINGLE] (*Numerator*) Sticks of Rope (*Denominator*)

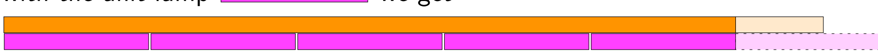
and therefore, as we saw in **??**, by the decimal phrase

7583. SINGLE (Numerator) Sticks of Rope (Denominator)

3. However, it is most important to realize that, no matter what *unit lump* we choose, the chances that a *given lump* will contain an *exact* number of *unit lumps* are strictly zero and that there will *always* be some **remainder**.

EXAMPLE 3.3.4. When we measure the given real world *lump* of ***rope***,

with the unit lump we get



where is the *remainder*.

3.2 Real Number Phrases

What makes *lumps of substance* more difficult to represent than *collections of items* is that we have to deal with the *remainder*

1. There is no point representing a lump of substance by the number phrase that represents its *measured part* if the remainder is not small enough that we can ignore it. But when the remainder *is* small enough that we can ignore it, then we will represent the remainder with the symbol [**...**].

EXAMPLE 3.3.5. Suppose we have a lump whose *measured part* is represented by

83 (Numerator) Sticks of Rope (Denominator)

and suppose that the *remainder* is small enough for our purpose. Then we will represent the remainder by

[...] (Numerator) Sticks of Rope (Denominator)

so that we can represent the lump by

sub-unit lump
real numerator

$$\underbrace{83}_{\text{Numerator}}\ \underbrace{\text{Sticks of Rope}}_{\text{Denominator}} + \underbrace{[...]}_{\text{Numerator}}\ \underbrace{\text{Sticks of Rope}}_{\text{Denominator}}$$

2. When the remainder is *not* small enough that we can ignore it, then what we do is to measure just the remainder with a **sub-unit lump**, that is a smaller lump such that **Ten** of these smaller lumps amount to the same as the *unit-lump*. And in fact the metric prefix **Deci** which we saw in **??** provides us automatically with a name for this sub-unit lump.

Example 3.3.6. Suppose we have a lump whose *measured part* is represented by

$$\underbrace{83}_{\text{Numerator}}\ \underbrace{\text{Sticks of Rope}}_{\text{Denominator}}$$

and suppose that the remainder is in fact too large to be represented by [...].
In that case, we measure the *remainder* with the *sub-unit length* TEN of which amount to the same as the original unit-length and whose name will therefore be DeciSticks of Rope and we might then be able to represent the given lump by, for instance,

83 Sticks of Rope + 7 DeciSticks of Rope + [...] DeciSticks of Rope

where [...] *might* now represent a remainder "too small to matter".
If the remainder is still *not* small enough not to matter, then we just repeat the above maneuver with a *sub-sub-unit lump* and we might get something like

83 Sticks of Rope + 7 DeciSticks of Rope + 4 CentiSticks of Rope + [...] CentiSticks of Rope

where [...] *might* now represent a remainder "too small to matter".
If not, etc.

3. What we will usually do is to use *decimal phrases* to represent the *measured part*.

Example 3.3.7. In the previous example, using *decimal phrases* for the measured parts, we would write:

$$\underbrace{83.}_{\text{Numerator}}\ \underbrace{\text{Sticks of Rope}}_{\text{Denominator}}$$

$$\underbrace{83.7}_{\text{Numerator}}\ \underbrace{\text{Sticks of Rope}}_{\text{Denominator}}$$

$$\underbrace{83.74}_{\text{Numerator}}\ \underbrace{\text{Sticks of Rope}}_{\text{Denominator}}$$

4. We will use **real numerators**, that is decimal phrase followed by [...], to represent the given lump itself.

EXAMPLE 3.3.8. In the previous example, using *real numerators*, we would write for the given length of rope:

$$\underbrace{83. + [...]}_{\text{Numerator}}\ \underbrace{\text{Sticks of Rope}}_{\text{Denominator}}$$

$$\underbrace{83.7 + [...]}_{\text{Numerator}}\ \underbrace{\text{Sticks of Rope}}_{\text{Denominator}}$$

$$\underbrace{83.74 + [...]}_{\text{Numerator}}\ \underbrace{\text{Sticks of Rope}}_{\text{Denominator}}$$

3.3 The U.S. Customary System of Units

In the United States, though, as we will now see, the *units of stuff* are set by the U.S. Customary System of Units (See http://www.ndt-ed.org/GeneralResources/Units/USCustomarySystem.htm) which has none of the features we used above.

1. The bundling rates do not stay the same as we keep on bundling at higher levels.

EXAMPLE 3.3.9. The U.S. Customary System of Units sets the bundling rates for *lengths* as:

Bundle	Single item
Bundle of TWELVE ***inches***	ONE ***foot***
Bundle of THREE ***feet***	ONE ***yard***
Bundle of SIX HUNDRED SIXTY ***yards***	ONE ***mile***

2. The bundling rates are different depending on the *kind* of stuff being dealt with.

EXAMPLE 3.3.10. The U.S. Customary System of Units sets the bundling rates for *liquids* as:

Bundle	Single item
Bundle of EIGHT ***ounces***	ONE ***cup***
Bundle of TWO ***cups***	ONE ***pint***
Bundle of TWO ***pints***	ONE ***quart***
Bundle of FOUR ***quarts***	ONE ***gallon***

but sets the bundling rates for *weights* as metric system

Bundle	Single item
Bundle of SIXTEEN ***ounces***	ONE ***pound***
Bundle of HUNDRED ***pounds***	ONE ***hundredweight***
Bundle of TWENTY ***hundredweight***	ONE ***ton***

3. It is often impossible to find a unit of stuff that is *commensurate* with the amount of stuff to be measured.

EXAMPLE 3.3.11.

- A ***mile***, the largest U.S. Customary Unit of length, is still not large enough to be commensurate with, say, the distance between the sun and the star Alpha Centauri because that would give 25689592880812.363 Miles which is not really informative.
- An ***inch***, the smallest U.S. Customary Unit of length, is still not small enough to be *commensurate* with the distance between two atoms because we would get 0 **inches** $+$ [...] which is not really informative either.

4. The units can be too far apart which then makes the approximation difficult to control.

EXAMPLE 3.3.12. The jump from ***Yard*** to ***Mile*** is very large which can make dealing with remainders difficult. When measuring the length of, say, a small road, we may run into the following catch-22 situation:

- If we use ***miles*** as units of length, the remainder can be up to ONE ***mile*** which for a small road may be too much because a small road may not even be ONE ***mile*** long in which case the remainder would be all there is and therefore certainly not too-small-to-matter-here.
- If we use ***yards*** as units of length, this allows the remainder to be up to ONE ***yard*** which, even for a small road, may be too small because the number of ***yards*** is likely to depend on how the road was surveyed and thus the remainder does not really correspond to the approximation.

The U.S. Customary System of Units does provide units of length between ***yards*** and ***miles*** such as ***furlong*** (See http://en.wikipedia.org/wiki/Furlong) but nobody seems to use ***furlongs*** anymore. For instance, interstate roads are marked in ***tenths of a mile*** rather than in ***furlongs***.

5. In fact, the U.S.A., Liberia and Myanmar (AKA Burma) are the only countries to use the U.S. System of Customary Units. All other countries—as well as the scientific and military worlds in the U.S.—use the International System of Units (AKA). (See http://en.wikipedia.org/wiki/Metrication_in_the_United_States.)

3.4 The International System of Units

Contrary to the U.S. Customary System of Units, the International System of Units does *not* provide a list of units because there would then be in that list both a largest unit with no guarantee that this largest unit would always be large enough and a smallest unit with no guarantee that this smallest unit would always be small enough.

base unit
base␣unit,␣fundamental
base␣unit,␣derived

What the International System of Units does is to provide a mechanism for creating and naming *systematically* an infinite number of units, no matter what the kind of stuff, including, in any particular situation, units as large as we need and units as small as we need. (See http://en.wikipedia.org/wiki/International_System_of_Units.)

1. More precisely, the International System of Units specifies a single of stuff for each kind of stuff but the way it does it depends on the kind of stuff:

i. For each one of the following seven kinds of stuff, the International System of Units specifies the base unit by way of a *real-world process* (See for instance http://en.wikipedia.org/wiki/Metre):

Kind of stuff	Base Unit
Length	meter
Mass	gram
Time	second
Electric current	ampere
Temperature	kelvin
Luminous Intensity	candela
Amount Of Substance	mole

We will refer to these base units as **fundamental base units**.[2]

ii. For all the other substances, the International System of Units specifies a base unit, referred to here as **derived base unit**, by way of a *paper-world procedure* involving only multiplication and division of any of the seven fundamental base units and/or any of the base units that have already been derived.

EXAMPLE 3.3.13. Since the area of a rectangle is its length times its width, the (derived) base unit for areas is derived from the (fundamental) base unit for length as meter×meter AKA square meter

EXAMPLE 3.3.14. Since we get the average speed of a vehicle traveling a distance by dividing that distance by the time it took the vehicle to travel that distance, the (derived) base unit for speed is meter/second.

[2]Educologists are of course well aware that the law adopted by the National Constituting Assembly of the French Revolution on March 30, 1791 mandated a "natural" definition of the primary units, that is a definition that would be neither anthropocentric nor nationalistic.

secondary␣units

2. In order to create **secondary units** from the base unit, the International System of Units uses the *metric prefixes* which we already saw in **??**:

Kilo	Hecto	Deka	—	Deci	Centi	Milli

which work exactly the same as

Thousand	Hundred	Ten	Single	Tenth	Hundredth	Thousandth

Example 3.3.15. When the substance is *length*, then the base unit is Meter and secondary units are

Kilo	Hecto	Deka		Deci	Centi	Milli
Meter	Meter	Meter	Meter	Meter	Meter	Meter

so that, for instance, instead of writing 32.6 Meters we can write:

0.0326 KiloMeters,
0.326 HectoMeters,
3.26 DekaMeters,

326. DeciMeters,
3260. CentiMeters,
32600. MilliMeters,

Index