Chapter 4
Comparisons

Comparing Collections, 1 • Comparison Sentences: Equalities and Strict Inequalities, 2 • Comparison Sentences: Weak Inequalities, 5 • Comparison Sentences For Large Collections, 7 • Graphing Sets of Number Phrases, 8 • Meeting A Requirement, 9.

4.1 Comparing Collections

Two real world collections can compare in three different ways depending on whether, after we have done a one-to-one matching, there are leftover items in the first collection or in the second collections or no leftover item whatsoever:

• When the leftover items are in the first collection, we will say that the first collection is larger than the second collection.

**Example 4.4.1.** To compare Jack’s with Jill’s in the real world, we match Jack’s collection one-to-one with Jill’s collection:

and we have that

Jack’s collection is larger than Jill’s collection

• When the leftover items are in the second collection, we will say that the
Chapter 4. Comparisons

is smaller than
is the same as
comparison sentences
<
is smaller than
is less than

first collection is smaller than the second collection.

**Example 4.4.2.** To compare Jack’s first collection with Jill’s second collection in the real world, we match Jack’s collection one-to-one with Jill’s collection:

and we have that

Jack’s collection is smaller than Jill’s collection

• When there is no leftover item, we will say that the first collection is the same in size as the second collection.

**Example 4.4.3.** To compare Jack’s first collection with Jill’s second collection in the real world, we match Jack’s collection one-to-one with Jill’s collection:

and we have that

Jack’s collection is the same as Jill’s collection

4.2 Comparison Sentences: Equalities and Strict Inequalities

1. Given two real world collections and the number phrases that represent then, we now want to write on paper comparison sentences, that is sentences (true or false) that state whether the first collection is larger than, smaller than or the same as the second collection. For that we introduce the following comparison verbs:

• We will use the comparison verb < to represent the comparison is smaller than and we will read < either as is smaller than or as is less than.

**Example 4.4.4.** Given Jack’s first collection and Jill’s second collection, the comparison sentence which states that

Jack’s collection is smaller than Jill’s collection
4.2. Comparison Sentences: Equalities and Strict Inequalities

is  
\[ 9 \text{ Dollars} < 5 \text{ Dollars} \]

• We will use the comparison verb \( > \) to represent the relationship is larger than and we will read \( > \) either as is larger than or as is more than.

**Example 4.4.5.** Given Jack’s \[
\begin{array}{c}
\text{money}
\end{array}
\] and Jill’s \[
\begin{array}{c}
\text{money}
\end{array}
\] the comparison sentence which states that Jack’s collection is larger than Jill’s collection is
\[ 5 \text{ Dollars} > 3 \text{ Dollars} \]

• We will use the comparison verb \( = \) to represent the comparison is the same as and we will read \( = \) as is equal to.

**Example 4.4.6.** Given Jack’s \[
\begin{array}{c}
\text{money}
\end{array}
\] and Jill’s \[
\begin{array}{c}
\text{money}
\end{array}
\] the comparison sentence which states that Jack’s collection is the same as Jill’s collection is
\[ 3 \text{ Dollars} = 9 \text{ Dollars} \]

**Note.** We use the same words for the real-world comparisons and for the comparison verbs we write on paper because, even though we need to distinguish the real-world from the paper-world, that is the way it goes and, anyway, nobody could remember which is to be used when talking about the real world and which is to be used when talking about the paper world. So, we will have to use other ways to keep in mind whether we are working in the real world or in the paper world.

2. Sentences involving the comparison verbs \( > \) or \( < \) are called strict inequalities while sentences involving the verb \( = \) are called equalities.

**Example 4.4.7.**
\[ 13 \text{ Dollars} < 7 \text{ Dollars} \quad \text{and} \quad 8 \text{ Dollars} > 2 \text{ Dollars} \quad \text{are strict inequalities} \]
\[ 3 \text{ Dollars} = 3 \text{ Dollars} \quad \text{is an equality} \]

3. In order to be able to decide whether a comparison sentence is TRUE or FALSE, we first need to extend the concept of counting to counting from any start-number to any end-number so that we have to be able to count in either one of two directions, namely count up or count down. Either way, though

i. We start counting after the start-number. (Just as we started counting after 0.)

ii. We stop counting after the end-number.
**Example 4.4.8.** To count from the start-number 3 to the end-number 7, we start counting up after 3, that is 4, and stop after 7:

\[ 4, 5, 6, 7 \]

**Example 4.4.9.** To count from the start-number 37 to the end-number 12, we start counting down after 37, that is at 36, and stop after 12:

\[ 36, 35, 34, \ldots, 14, 13, 12 \]

4. While the real-world real-world process of comparing two real-world collections by matching one-to-one the two collections can be very painful with large collections, once the collections are represented by number-phrases, getting the TRUE comparison sentence is easy:

- If we need to count up, then the TRUE sentence must use the comparison verb `<`
- If we need to count down, then the TRUE sentence must use the comparison verb `>`
- If we don’t need to count at all, then the TRUE sentence must use the comparison verb `=

**Example 4.4.10.** The sentence

\[ 5 \text{ Dollars} < 3 \text{ Dollars} \]

is FALSE because to count from 5 to 3 we must count down. (And, indeed, in the real world, after a one-to-one matching of \(5\) and \(3\), the leftover items would be in the first collection.)

**Example 4.4.11.** The sentence

\[ 5 \text{ Dollars} > 3 \text{ Dollars} \]

is TRUE because to count from 5 to 3 we must count down. (And, indeed, in the real world, after a one-to-one matching of \(5\) and \(3\), the leftover items would be in the first collection.)

**Example 4.4.12.** The sentence

\[ 5 \text{ Dollars} = 3 \text{ Dollars} \]

is FALSE because to count from 5 to 3 we must count. (And, indeed, in the real world, after a one-to-one matching of \(5\) and \(3\), there would be leftover items.)
4.3 Comparison Sentences: Weak Inequalities

1. Most of the time, though, we do not want to speak FALSE and we use:

AGREEMENT 4.1. When no indication of TRUE or FALSE is given, comparison sentences will always be understood to be TRUE and this will go without saying.

So, instead of writing a comparison sentence that is FALSE and having to say that it is FALSE, we will write the negation of the FALSE comparison sentence which will be a TRUE comparison sentence whose truth will therefore “go without saying”.

To write the negation of a comparison sentence, we just slash the comparison verb in the comparison sentence.

**Example 4.4.13.** Instead of writing
\[ 5 \text{ Washingtons} < 3 \text{ Washingtons} \] is FALSE
we write
\[ 5 \text{ Washingtons} \not< 3 \text{ Washingtons} \]
which we read as
FIVE Washingtons is no less than THREE Washingtons.
and where the fact that it is TRUE goes without saying.

**Example 4.4.14.** Instead of writing
\[ 5 \text{ Washingtons} > 7 \text{ Washingtons} \] is FALSE
we write
\[ 5 \text{ Washingtons} \not> 7 \text{ Washingtons} \]
which we read as
FIVE Washingtons is no more than SEVEN Washingtons.
and where the fact that it is TRUE goes without saying.

**Example 4.4.15.** Instead of writing
\[ 3 \text{ Washingtons} = 7 \text{ Washingtons} \] is FALSE
we write
\[ 3 \text{ Washingtons} \neq 7 \text{ Washingtons} \]
which, naturally, we read as
FIVE Washingtons is not equal to SEVEN Washingtons.
and where the fact that it is TRUE goes without saying.

2. The comparison verbs \(<, >\), and \(=\) are mutually exclusive: given three comparison sentences involving the same number-phrases and the comparison verbs is larger than, is smaller than, and is the same as, as soon as we know that one of the three comparison sentences is TRUE, we automatically know that the other two comparison sentences must be FALSE.
EXAMPLE 4.4.16. Once we know that $5 \text{ Dollars} > 3 \text{ Dollars}$ is TRUE, we automatically know that both $5 \text{ Dollars} < 3 \text{ Dollars}$ and $5 \text{ Dollars} = 3 \text{ Dollars}$ must be FALSE.

Another way to put it is that as soon as we know that one of the three comparison sentences is FALSE, then one of the other two comparison sentences must be TRUE.

EXAMPLE 4.4.17. Once we know that $4 \text{ Dollars} > 7 \text{ Dollars}$ is FALSE, we automatically know that one of $4 \text{ Dollars} < 7 \text{ Dollars}$ and $4 \text{ Dollars} = 4 \text{ Dollars}$ must be TRUE.

3. In fact, while mathematicians do not mind writing negation of equalities, they don’t like writing negations of strong inequalities and prefer to make use of the observation we just made that the three comparison verbs, $<, >,$ and $=$, are mutually exclusive: if a strong inequality is FALSE, then either the other strong inequality or the equality must be TRUE. So, they prefer to write weak inequalities that say just that. More precisely,

- Instead of writing a comparison sentence with the verb $\neq$, read as “no more than”, we actually write a weak inequality with the comparison verb $\leq$, read as “less than or equal to”.

EXAMPLE 4.4.18. Instead of writing

$$3 \text{ Washingtons} \neq 7 \text{ Washingtons}$$

read as

THREE Washingtons is no more than SEVEN Washingtons.

mathematicians prefer to write “the other two” comparison sentences

$$3 \text{ Washingtons} < 7 \text{ Washingtons} \text{ or } 3 \text{ Washingtons} = 7 \text{ Washingtons}$$

in the shape of the weak inequality

$$3 \text{ Washingtons} \leq 7 \text{ Washingtons}$$

which, naturally, we read as

THREE Washingtons is less than or equal to SEVEN Washingtons.

and where the fact that it is TRUE goes without saying.

- Instead of writing a comparison sentence with the verb $\neq$, read as “no less than”, we actually write a weak inequality with the comparison verb $\geq$, read as “more than or equal to”.

EXAMPLE 4.4.19. Instead of writing

$$5 \text{ Washingtons} \neq 3 \text{ Washingtons}$$

read as

FIVE Washingtons is no less than THREE Washingtons.

mathematicians prefer to write “the other two” comparison sentences

$$5 \text{ Washingtons} \geq 3 \text{ Washingtons}$$

which, naturally, we read as

FIVE Washingtons is more than or equal to THREE Washingtons.

and where the fact that it is TRUE goes without saying.

Finally, the word inequality will cover both strong inequalities and weak
4.4 Comparison Sentences For Large Collections

In order to compare large collections, we represent them by decimal number-phrases but, both on paper and in our mind, we use in fact tabular number-phrases under a single heading as this by-passes the issue of whether or not the two decimal number-phrases have the same select-denominator.

There are two cases:

1. The leftmost digit of the tabular number-phrases are under different denominators. Then, regardless of the other digits, we can write either that:
   - the number-phrase with the leftmost digit left of the other leftmost digit is larger than the other number-phrase.
   or that
   - the number-phrase with the leftmost digit right of the other leftmost digit is smaller than the other number-phrase.

**Example 4.4.20.** Jack’s collection is represented by 5.87 DekaWashingtons and Jill’s collection is represented by 2341.6 DeciWashingtons.

i. The tabular number-phrases are:

<table>
<thead>
<tr>
<th></th>
<th>Kilo Washingtons</th>
<th>Hecto Washingtons</th>
<th>Deka Washingtons</th>
<th>Deci Washingtons</th>
<th>Centi Washingtons</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jack</td>
<td></td>
<td>5</td>
<td>8</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>Jill</td>
<td></td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

ii. Since Jill has HectoWashingtons while Jack doesn’t have any, and even though she has fewer DekaWashingtons, fewer Washingtons and fewer DeciWashingtons than Jack, we can write that

5.87 DekaWashingtons < 2341.6 DeciWashingtons

or that

2341.6 DeciWashingtons > 5.87 DekaWashingtons

2. The leftmost digit of the tabular number-phrases are under the same denominator. Then, regardless of the other digits, we can write either that:
   - the number-phrase with the larger leftmost digit is larger than the other number-phrase
   or that
   - the number-phrase with the smaller leftmost digit is smaller than the other number-phrase

**Example 4.4.21.** Jack’s collection is represented by 54.37 DekaWashingtons and Jill’s collection is represented by 3689.2 DeciWashingtons.

i. The tabular number-phrases are:
4.5 Graphing Sets of Number Phrases

Once we have represented a collection with a number phrase, we will often also want to graph that number phrase.

1. For that purpose, we will use rulers which are straight lines with a few labels:
   - an arrowhead to indicate the way numerators go up,
   - tick marks to be labeled with numerators,
   - the denominator next to the arrowhead.

   **AGREEMENT 4.2.** Rulers will always have to include an origin, that is a tickmark labeled 0.

**EXAMPLE 4.4.22.** Here is a ruler for number phrases with Washington as denominator:

2. Then, we will graph a number phrase with a dot which will be either:
   - A solid dot ⬤ to graph a numerator which IS in the set.
   - A hollow dot ○ to graph a numerator which IS NOT in the set.

**EXAMPLE 4.4.23.** The graph for the number phrase 3 Washingtons could be either:

- if the number phrase 3 IS in the set we want to graph,
- or
- if the number phrase 3 IS NOT in the set we want to graph.
4.6 Meeting A Requirement

1. Very often, we have to solve comparison problems, that is given a collection of items, called the data set, we need to get those items which meet a requirement given in the shape of a specifying form, that is a sentence with an empty box in which to fill in names of items. Those items whose name turn the specifying form into a TRUE sentence are called solutions and make up the solution subset for the given comparison problem.

Example 4.4.24. Given the data set
{George W. Bush, Hillary Clinton, John Kennedy, Barack Obama}
and the specifying form

is a past President of the United States.

• Since the sentence

is TRUE, George W. Bush is a solution.

• Since the sentence

is FALSE, Hillary Clinton is not a solution.

• Etc

So, the solution subset of the comparison problem is:
{George W. Bush, John Kennedy}

2. Usually, instead of using comparison forms, we will use comparison formulas that is, instead of a box, we will use:
• the letter $x$ to work as an unspecified numerator.
followed by
• the denominator of the number phrases in the data set,

Example 4.4.25. Instead of writing the comparison form

we will write the comparison formula


3. However, in order to focus, we will usually declare the denominator up front and then we will deal with just the numerators.

Example 4.4.26. Instead of writing the comparison form

we will write the comparison formula


comparison problem
data set
requirement
specifying form
box
solutions
solution subset
comparison formula
unspecified numerator
declare
we declare that the denominator is \textbf{Apples} and the comparison formula is:
\[
\begin{array}{c}
\text{\cellcolor{yellow}2} \\
> \\
\text{\cellcolor{blue}5}
\end{array}
\]

Declaring the denominator makes writing comparison problems a lot simpler.

\textbf{Example 4.4.27.} After declaring that the denominator is \textbf{Apples}, we just write:
- Data set: \{3, 4, 5, 6, 7, 8\}
  which we can graph as:
  
  \begin{center}
  \begin{tikzpicture}
    \draw[->] (0,0) -- (10,0);
    \foreach \x in {0,1,...,9} { \draw[fill=black] (\x,0) circle (2pt); }
    \filldraw[fill=yellow] (2,0) circle (2pt) node[above]{2};
    \node[below] at (10,0) {Apples};
  \end{tikzpicture}
  \end{center}
- Specifying formula: \( x \geq 5 \)
- Solution subset: \{5, 6, 7, 8\}
  which we can graph as:
  
  \begin{center}
  \begin{tikzpicture}
    \draw[->] (0,0) -- (10,0);
    \foreach \x in {0,1,...,9} { \draw[fill=black] (\x,0) circle (2pt); }
    \filldraw[fill=black] (5,0) circle (2pt) node[above]{5};
    \filldraw[fill=black] (6,0) circle (2pt); \filldraw[fill=black] (7,0) circle (2pt); \filldraw[fill=black] (8,0) circle (2pt);
    \node[below] at (10,0) {Apples};
  \end{tikzpicture}
  \end{center}

4. In this course, we will limit ourselves to basic comparison problems, that is, given a data set of number phrases, we will just compare them to a gauge number phrase.

\textbf{Example 4.4.28.} Find the solution subset of the comparison problem where the denominator is \textbf{Apples} and whose:
- Data set is \{3, 4, 5, 6, 7, 8\}
  which we can graph as:
  
  \begin{center}
  \begin{tikzpicture}
    \draw[->] (0,0) -- (10,0);
    \foreach \x in {0,1,...,9} { \draw[fill=black] (\x,0) circle (2pt); }
    \filldraw[fill=black] (3,0) circle (2pt); \filldraw[fill=black] (4,0) circle (2pt); \filldraw[fill=black] (5,0) circle (2pt); \filldraw[fill=black] (6,0) circle (2pt); \filldraw[fill=black] (7,0) circle (2pt); \filldraw[fill=black] (8,0) circle (2pt);
    \node[below] at (10,0) {Apples};
  \end{tikzpicture}
  \end{center}
- Comparison formula is \( x \geq 5 \)
  where 5 is the gauge numerator.

However, we will emphasize the distinction between:
- \textbf{Inequations}\footnote{Although supposedly concerned with the relevance of mathematics to the “real world”, Educologists are strangely indifferent to the fact that, \textit{in real life}, inequations are vastly more prevalent than equations. Not to mention that mixing up (in)\textit{equations} and (in)\textit{equalities} cannot help.} which are comparison formulas with any one of the five comparison verbs: \( > \quad < \quad \neq \quad \leq \quad \geq \)
- \textbf{Equations} which are comparison formulas with the comparison verb =

5. Finding the solution subset of basic comparison problem is straightforward: We just try each number phrase in the data set.

\textbf{Example 4.4.29.} We declare that we are dealing with \textbf{Apples}. Given the data set \{3, 4, 5, 6, 7, 8\},
which we can graph as:

\begin{center}
\begin{tikzpicture}
\draw[->] (0,0) -- (10,0);
\foreach \x in {0,1,...,9} { \draw[fill=black] (\x,0) circle (2pt); }
\filldraw[fill=black] (3,0) circle (2pt); \filldraw[fill=black] (4,0) circle (2pt); \filldraw[fill=black] (5,0) circle (2pt); \filldraw[fill=black] (6,0) circle (2pt); \filldraw[fill=black] (7,0) circle (2pt); \filldraw[fill=black] (8,0) circle (2pt);
\node[below] at (10,0) {Apples};
\end{tikzpicture}
\end{center}

and the comparison formula
we try each numerator in the data set:

\[ \begin{align*}
3 \geq 5 & \text{ which is FALSE,} \\
4 \geq 5 & \text{ which is FALSE,} \\
5 \geq 5 & \text{ which is TRUE,} \\
6 \geq 5 & \text{ which is TRUE,} \\
7 \geq 5 & \text{ which is TRUE,} \\
8 \geq 5 & \text{ which is TRUE,}
\end{align*} \]

So, the solution subset is 
\{5, 6, 7, 8\}
that is:

5 Apples, 6 Apples, 7 Apples, 8 Apples

which we can graph as:
Index

<, 2
=, 3
>, 3
\geq, 6
\leq, 6

arrowhead, 8

basic comparison problems, 10
box, 9

compare, 1
comparison formula, 9
comparison problem, 9
comparison sentences, 2
count down, 3
count up, 3
data set, 9
declare, 9
direction, 3

direction, 3
equality, 3
equation, 10
gauge number phrase, 10
graph, 8

Hollow dot, 8

inequality, 6
inequation, 10
is equal to, 3

is larger than, 1, 3
is less than, 2
is more than, 3
is smaller than, 2
is the same as, 2

label, 8
leftover item, 1

mutually exclusive, 5

negation, 5

one-to-one matching, 1
origin, 8

requirement, 9
ruler, 8

slash, 5
solid dot, 8
solution subset, 9
solutions, 9
specifying form, 9
start-number, 3
strict inequality, 3

tick mark, 8

unspecified numerator, 9

weak inequality, 6