Chapter 6

Multiplication - Division

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Multiplication and division are very different from addition and subtraction in several different ways.

6.1 Can Money Be Multiplied By Money?

A major way in which *multiplication* differs from both *addition* and *subtraction* is that while we could add and/or subtract number phrases (with a common denominator) and get as a result a number phrase (with that same common denominator), we usually cannot multiply number phrases and get as a result a number phrase.

Example 6.1. Given the number phrase 7 **Dimes**, we can add or subtract the number phrase 2 **Dimes** to represent attaching or detaching a collection but what could multiplying 7 **Dimes** by 3 **Dimes** possibly represent?

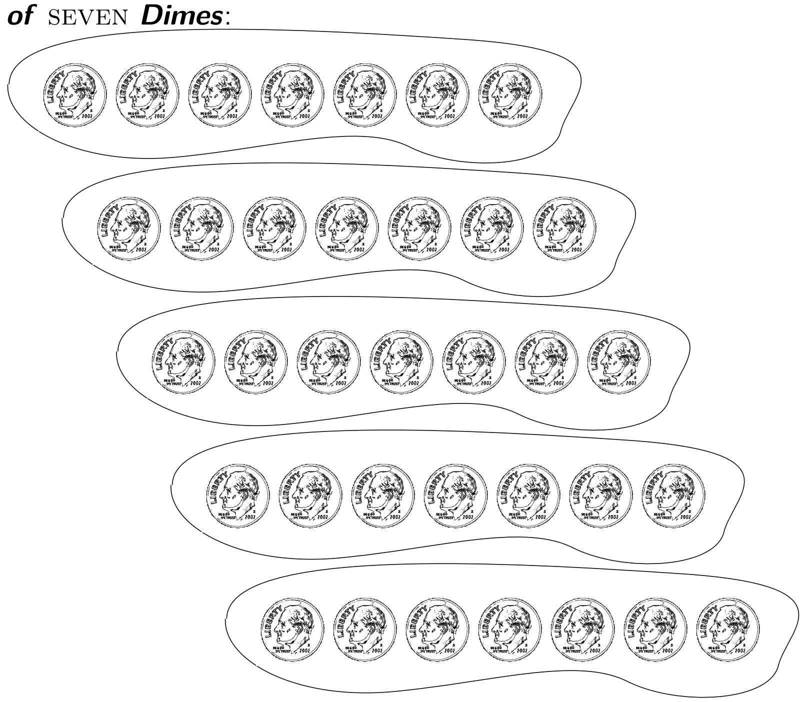
6.2 Repeated Addition

A number phrase in which the *denominator* is itself a *number phrase* represents a *collection of collections*.

repeated addition of number phrases
multiply

EXAMPLE 6.2. The number phrase $7\,\textbf{Dimes}$ represents the collection

and the number phrase $5[7\,\textbf{Dimes}]$ represents a collection of FIVE ***collections of*** SEVEN ***Dimes***:



We get the *total number of items* by **repeated addition of number phrases**, that is by adding the number phrases which represent the identical collections being collected. But then it is usual to say that this is what **multiplying** the number of collections being collected by the number of items in the identical collections being collected does.

EXAMPLE 6.3. The total number of ***Dimes*** in the previous example is represented by:

$$\begin{aligned} 7\,\textbf{Dimes} + 7\,\textbf{Dimes} + 7\,\textbf{Dimes} + 7\,\textbf{Dimes} + 7\,\textbf{Dimes} &= [7+7+7+7+7]\,\textbf{Dimes} \\ &= 35\,\textbf{Dimes} \end{aligned}$$

and then it is usual to that this is

$$= [5 \times 7]\,\textsf{Dimes}$$

However:

i. 5×7 gets the result of a *repeated addition of number phrases* but, as already mentioned at the outset, does *not* get the result of a multiplication of number phrases the way addition and subtraction are addition and subtraction *of number phrases*.

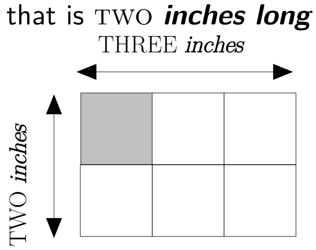
ii. Moreover, even looking at multiplication as replacing repeated addition only works *when the numerator is a* counting number but this does not extend to when the numerator is a *decimal number*.

EXAMPLE 6.4. While $5[7\,\textsf{Dimes}]$ represents a collection of FIVE ***collections of*** SEVEN ***Dimes***, what collection of ***collections of*** SEVEN ***Dimes*** could $5.0384[7\,\textsf{Dimes}]$ possibly represent?

6.3 Computing Areas

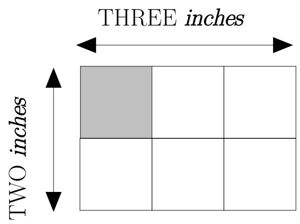
In certain rare cases, a multiplication *of number phrases* does make sense but the denominator of the *result* is *different* from the denominator of the original number phrases.

EXAMPLE 6.5. $2\,\textsf{Inches} \times 3\,\textsf{Inches} = [7 \times 3]\,\textsf{SquareInches}$
which represents the *area* of a ***two-by-three rectangle***, that is a **rectangle** that is TWO ***inches long*** one way and THREE ***inches long*** the other way:



Indeed, if we want to tile this rectangle with ***one-inch-by-one-inch mosaics*** we get

worth
unit-worth
exchange
exchange rate
co-number phrase
co-multiply



Counting the ***mosaics*** shows that we will need SIX ***one-inch-by-one-inch mosaics***.

But this type of multiplication *does* extend to *decimal numbers*.

6.4 Co-multiplication

We seldom deal with a collection without wanting to know what the (money?) **worth** of the collection is, that is how much money the collection could be exchanged for.

EXAMPLE 6.6. Given a collection of FIVE ***apples***, and given that the *worth* of an ***apple*** is SEVEN ***cents***, the real-world *process* for finding the *worth* of the collection is to exchange each ***apple*** for SEVEN ***cents***. Altogether, we end up exchanging the whole collection for THIRTY-FIVE ***cents*** which is therefore the total *worth* of the collection.

1. A **unit-worth** of a given substance is the amount of another kind of substance we can **exchange** for one unit of the given substance. The real world **exchange rate** is then represented on paper by a **co-number phrase** in the shape of a "fraction".

EXAMPLE 6.7. Let the substance be ***Gasoline***. Then, if we can exchange each ***Gallons of Gas*** for 3.149 **Dollars**, we will represent this *exchange rate* by the co-number phrase $3.149\,\frac{\text{Dollars}}{\text{Gallon of Gas}}$ which we read 3.149 **Dollars** per **Gallon of Gas**

2. We obtain the worth of an amount of substance at a given exchange rate by **co-multiplying** the number phrase that represents the amount of substance by the co-number phrase that represents the unit worth of that substance. And of course exactly the same goes for collections of items.
The procedure for co-multiplying is quite simple:

percentage

i. multiply the *numerators*
ii. multiply the *denominators* with cancellation.

EXAMPLE 6.8. Given a collection of FIVE ***apples*** and given that the *worth* of ONE ***apple*** is SEVEN ***cents***, the real-world *process* for finding the *worth* of the collection is to exchange each ***apple*** for SEVEN ***cents***. Altogether, we end up exchanging the whole collection for THIRTY-FIVE ***cents*** which is therefore the total *worth* of the collection.
On paper, we write

$$\begin{aligned} 5\ \textbf{Apples} \times 7\ \frac{\textbf{Cents}}{\textbf{Apple}} &= (5 \times 7) \left(\cancel{\textbf{Apples}} \times \frac{\textbf{Cents}}{\cancel{\textbf{Apple}}}\right) \\ &= 35\ \textbf{Cents} \end{aligned}$$

3. Co-multiplication is at the heart of a part of mathematics called DIMENSIONAL ANALYSIS (See https://en.wikipedia.org/wiki/Dimensional_analysis) that is much used in sciences such as PHYSICS, MECHANICS, CHEMISTRY and ENGINEERING where people have to "cancel" denominators all the time.

EXAMPLE 6.9.

$$5\ \textbf{Hours} \times 7\ \frac{\textbf{Miles}}{\textbf{Hour}} = (5 \times 7) \left(\cancel{\textbf{Hours}} \times \frac{\textbf{Miles}}{\cancel{\textbf{Hour}}}\right) = 35\ \textbf{Miles}$$

EXAMPLE 6.10.

$$5\ \textbf{Square-Inches} \times 7\ \frac{\textbf{Pound}}{\textbf{Square-Inch}} = (5 \times 7) \left(\cancel{\textbf{Square-Inches}} \times \frac{\textbf{Pound}}{\cancel{\textbf{Square-Inch}}}\right) = 35\ \textbf{Pounds}$$

Co-multiplication is also central to a part of mathematics called LINEAR ALGEBRA that is itself of major importance both in many other parts of mathematics and for all sort of applications in sciences such as ECONOMICS.

More modestly, *co-multiplication* also arises in **percentage** problems:

EXAMPLE 6.11.

$$5\ \textbf{Dollars} \times 7\ \frac{\textbf{Cents}}{\textbf{Dollar}} = (5 \times 7) \left(\cancel{\textbf{Dollars}} \times \frac{\textbf{Cents}}{\cancel{\textbf{Dollar}}}\right) = 35\ \textbf{Cents}$$

assign
round
share
leftover

6.5 Sharing In The Real World

We first look at the *real-world process* and then we look at the corresponding *paper-world procedure*. In the real world, we often encounter situations in which we have to **assign** (equally) the items in a first collection to the items of another collection.

The *process* is to make **rounds** during each of which we *assign* one item of the first collection to each one of the items in the second collection. The process comes to an end when, after a round has been completed,

- there are items left unassigned but not enough to complete another round. The **share** is then the collection of items from the first collection that have been assigned to each item of the second collection and the **leftovers** are the collection of items from the first collection left unassigned after the process has come to an end.

EXAMPLE 6.12. In the real world, say we have a collection of seven dollar-bills which we want to assign to each and every person in a collection of three person. We want to know how many dollar-bills we will assign to each person and how many dollar-bills will be left-over.
i. We make a *first round* during which we hand-out one dollar-bill to each and every person in the collection. This uses three dollar-bills and leaves us with four dollar-bills after the first round.
ii. We make a *second round*, we hand-out one dollar-bill to each and every person in the collection. This uses another three dollar-bills and leaves us with one dollar-bill after the second round.
iii. If we try to make a *third round*, we find that we cannot complete the third round.
So, the *share* is two dollar-bills and the *leftovers* is one dollar-bill.

or,

- there is no item left unassigned. The *share* is again the collection of items from the first collection that have been assigned to each item of the second collection and there are no *leftovers*.

EXAMPLE 6.13. In the real world, say we have a collection of eight dollar-bills which we want to assign to each and every person in a collection of four person. We want to know how many dollar-bills we will assign to each person and how many dollar-bills will be left-over.
i. We make a *first round* during which we hand-out one dollar-bill to each

and every person in the collection. This uses four dollar-bills and leaves us with four dollar-bills after the first round.
ii. We make a *second round*, we hand-out one dollar-bill to each and every person in the collection. This uses another four dollar-bills and leaves us with no dollar-bill after the second round.
iii. So, we cannot make a *third round*.
So, the *share* is two dollar-bills and there are no leftovers.

division
dividend
divisor
quotient
remainder
trial and error
try
partial product

6.6 Division On Paper

The paper *procedure* that corresponds to the real-world process is called **division**.

1. *Division* will involve the following language:

- The number-phrase that represents the first collection, that is the collections of items *to be assigned* to the items of the second collection, is called the **dividend**,
- The number-phrase that represents the second collection, that is the collection of items *to which* the items of the first collection are to be assigned, is called the **divisor**,
- The number-phrase that represents the *share* is called the **quotient**,
- The number-phrase that represents the *leftovers* is called the **remainder**.

EXAMPLE 6.14. Given a real-world situation with a collection of eight dollar-bills to be assigned to each and every person in a collection of four persons,

- The *dividend* is 7 **Dollars**
- The *divisor* is 3 **Persons**
- The *quotient* is $2\frac{\textbf{Dollars}}{\textbf{Person}}$
- The *remainder* is 1 **Dollar**

2. The *division procedure* taught in elementary schools is a **trial and error** procedure which follows the real-world process closely inasmuch as each *round* is represented by a **try** in which:

i. We use the *multiplication procedure* to find the **partial product** which represents how many items *have been used* by the end of the corresponding *real-world round*.

partial remainder

ii. We use the *subtraction procedure* to find the **partial remainder** which represents how many items, if any, are *left over* by the end of the corresponding *real-world round*.

EXAMPLE 6.15. In order to divide 987 by 321, we go through the following *tries*:

First try:

i. We multiply the *divisor* 321 by 1 which gives the *partial product* 321:

$$\begin{array}{r} 1 \\ 321\;\big)\overline{987} \\ 321 \end{array}$$

ii. We subtract the *partial product* 321 from the *dividend* 987:

$$\begin{array}{r} 1 \\ 321\;\big)\overline{987} \\ \underline{321} \\ 666 \end{array}$$

which leaves the *partial remainder* 666 which is *larger* than 321 therefore too large.

Second try:

i. We multiply the *divisor* 321 by 2 which gives the *partial product* 642:

$$\begin{array}{r} 2 \\ 321\;\big)\overline{987} \\ 642 \end{array}$$

ii. We subtract the *partial product* 642 from the *dividend* 987:

$$\begin{array}{r} 2 \\ 321\;\big)\overline{987} \\ \underline{642} \\ 345 \end{array}$$

which leaves the *partial remainder* 345 which is *larger* than 321 therefore too large.

Third try:

i. We multiply the *divisor* 321 by 3 which gives the *partial product* 963:

$$\begin{array}{r|l} & 3 \\ \hline 321 \ ) & 987 \\ & 963 \end{array}$$

ii. We subtract the *partial product* 963 from the *dividend* 987:

$$\begin{array}{r|r} & 3 \\ \hline 321 \ ) & 987 \\ & 963 \\ \hline & 24 \end{array}$$

which leaves the *partial remainder* 24 which is *smaller* than 321 so this is it! **Fourth try** (Just to check that this is really it.)

i. We multiply the *divisor* 321 by 4 which gives the *partial product* 1284:

$$\begin{array}{r|l} & 4 \\ \hline 321 \ ) & 987 \\ & 1284 \end{array}$$

ii. We cannot subtract the *partial product* 1284 from the *dividend* 987:

$$\begin{array}{r|r} & 4 \\ \hline 321 \ ) & 987 \\ & 1284 \\ \hline \end{array}$$

and indeed we cannot complete the fourth try and must go back to the last complete try, that is the third try, and we get that the *quotient* is 3 and the *remainder* 24.

This procedure, though, has two severe shortcomings:

- All these *full multiplications* require a lot of work.
- This procedure will *not* extend to *polynomials*

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