Chapter 7

Signed Numbers – Comparisons

LANGUAGE 7.1. To make it clear which kind of collections we are talking about, from now on we will refer to the collections which were introduced in **??** as collections of **plain items** and to the number phrases which represent them as **plain number phrases**.

There are two issues with *plain* number phrases:

- While we can count *up* as far as we want we cannot always count *down* as far as we want.
- Number phrases can represent collections because all the items in a collection are of *one* kind but there are many situations in which items can come in either one of *two* kinds.

EXAMPLE 7.1.

- 3056.38 **Dollars** does not say if this was a *deposit* or a *withdrawal*,
- 37 800 **Dollars** does not say if a business is **in the red** (*owes* that money) or **in the black** (*has* that money).
- 62 **Dollars** does not say if a gambler is **ahead of the game** (has won more than s/he has lost) or **in the hole** (has lost more than s/he has won).

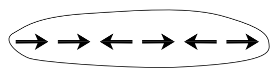
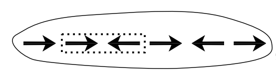
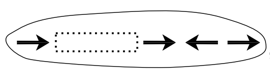
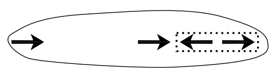
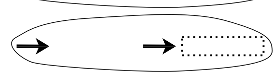
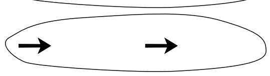
oriented items
orientation
opposite orientations
cancel
directed action
sided state
benchmark

– 2 **Feet** does not say if a point is to the *left* or to the *right* of a benchmark.
– 5 **Inches** does not say if a point is *above* or *below* a baseline.

We will now develop a symbolic system which can do all that.

7.1 Oriented Items

1. In the real world, there are many situations where we have to deal with collections of **oriented items**, that is items with *either one of two* **orientations** but where items with **opposite orientations cancel** each other so that collections of oriented items can only involve items that are all oriented the same way.

Example 7.2. The collection  reduces automatically to only items with the same orientation: , , , , .

2. In the real-world, *oriented items* generally fall into either one of two categories:

- Items called **directed actions** which are "moves" of one kind or another but that can go either in *this-direction* or *that-direction*.

  Example 7.3.
  – a businesswoman may *deposit* three thousand dollar on a bank account or may *withdraw* three thousand dollars from a bank account.
  – a gambler may *win* sixty-two dollars or may *lose* sixty-two dollars.
  – on a horizontal line, a point can be moved two feet *leftward* or two feet *rightward*
  – on a vertical line, a point can be moved five inches *upward* or five inches *downward*

- Items called **sided states** which can be either on *this-side* or *that-side* of some **benchmark**.

plain numerator
declare
standard direction
opposite direction
standard side
opposite side
signed numerator
sign
positive numerator

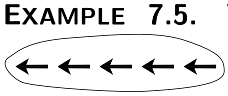
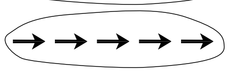
EXAMPLE 7.4.

– a business may be three thousand dollars *in the red* or three thousand dollars *in the black*
– a gambler may be sixty-two dollars *ahead of the game* or sixty-two dollars *in the hole*.
– on a horizontal line with some benchmark, a point may be two feet *to the left* of the benchmark or two feet *to the right* of the benchmark
– on a vertical line with some baseline, a point may be five inches *above* the benchmark or five inches *below* the benchmark

7.2 Signed Number Phrases

LANGUAGE 7.2. To make it clear which kind of numerator we are talking about, we will refer to the numerators which were introduced in Chapter 2 as **plain numerators**.

To *represent* collections of oriented items on paper, we could just use plain numerators and two denominators, one for each orientation.

EXAMPLE 7.5. We could represent the collection of oriented items  by the plain number phrase 5 **Left Arrows** and the collection  by the plain number phrase 5 **Right Arrows**

However, even though orientation is arguably a *qualitative* issue, representing the *orientation* as part of the *numerator* instead of as part of the *denominator* will enormously facilitate *computations*.

1. The first thing we have to do is to **declare**

- which direction is to be the **standard direction** and which direction is to be the **opposite direction**,
- which side of the benchmark is going to be the **standard side** and which side is to be the **opposite side**,

2. Then, a **signed numerator** will consists of:

- A **sign** for which, traditionally, we use + to indicate one orientation and – to indicate the other orientation. Numerators signed with + are called **positive numerators** and numerators signed with – are

negative numerator
opposite sign
size
signed number-phrase

called **negative numerators**. Thus, **opposite signs** represent opposite orientations.

- A **size** which is a *plain numerator*.

Example 7.6.

> **Note 7.1.** From now on, the symbol $+$ will thus be used for two purposes as $+$ can now be:
> - The *symbol* for addition of plain numerators as in Chapter 5,
> - The *sign* of a positive numerator as now.

In order to make it always clear for which purpose $+$ is being used, we use

> **Agreement 7.3.** In this text, $+$ as *sign* of a positive numerator will *never* "go without saying".

Indeed, with **Agreement 7.3** we will always know whether $+$ stands for the *symbol* for plain addition or for the *sign* of a positive numerator.

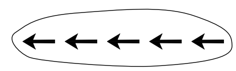
Example 7.7. In $2 + 5$, the symbol $+$ cannot be the sign of 5 because, by **Agreement 7.3**, 2 has to be a *plain* numerator and what could a *plain* numerator followed by a *signed* numerator possibly represent?

But the price for that is that if we want to write a *positive* numerator, we *must* write the sign $+$ as, otherwise, by **Agreement 7.3**, with no $+$, the numerator will be seen as being *plain*.

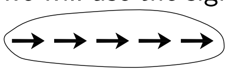
Example 7.8. If we want to talk about the opposite of -5, we *must* write $+5$ because if we write just 5 this will be seen as a *plain* numerator which is not the opposite of anything and has no opposite.

3. Then, a **signed number-phrase** will consist of:

- a signed-numerator
- a denominator

Example 7.9. We declare that $+$ will represent *right* steps and therefore that $-$ will represent *left* steps. Then, to represent the collection 

signed ruler

we will use the signed number phrase -5 **Arrows** and to represent the collection  we will use the signed number phrase $+5$ **Arrows**

EXAMPLE 7.10. We declare that the *standard* direction is to *win* money so that to *lose* money is the *opposite* direction. Then,

When a *real-world* gambler:	We write on *paper*:
• *wins* sixty-two dollars	$+62$ **Dollars**
• *loses* sixty-two dollars	-62 **Dollars**

in which $+62$ is a *positive* signed-numerator and -62 is a *negative* signed-numerator.

EXAMPLE 7.11. We declare that the *standard* side is *in-the-black* so that *in-the-red* is the *opposite* side. Then,

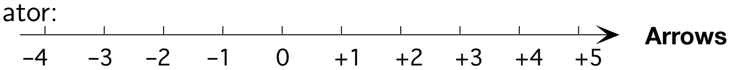
When a *real-world* business is:	We write on *paper*:
• three thousand dollars *in-the-black*	$+3000$ **Dollars**
• three thousand dollars *in-the-red*	-3000 **Dollars**

in which $+3000$ is a *positive* signed-numerator and -3000 is a *negative* signed-numerator.

7.3 Graphing Signed Number Phrases

1. To *graph* signed number phrases, we use **signed rulers**.

EXAMPLE 7.12. Here is a signed ruler for signed numerators with **Arrows** as denominator:



Just as with plain number phrases, we will use *solid dots* and *hollow dots* to graph signed number phrases.

2. From the *graphic* viewpoint:

- The *sign* of a signed numerator codes *which side* of 0 the graph of the signed numerator is.

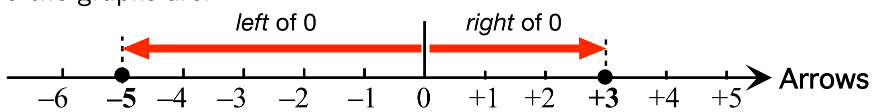
plain counting

EXAMPLE 7.13. Since

Sign of -5 $= -$, the signed numerator -5 is left of 0.

Sign of $+3$ $= +$, the signed numerator $+3$ is right of 0.

So the graphs are:

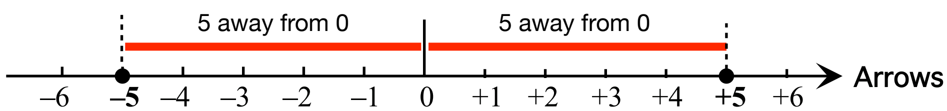


- The *size* of a signed numerator codes *how far away* from 0 the signed numerator is on a signed ruler.

EXAMPLE 7.14. Since

Size of -5 is 5, the signed numerator -5 is 5 away from 0,

Size of $+5$ is 5, the signed numerator $+5$ is 5 away from 0.



7.4 Signed Counting

1. In Chapter 2, we used **plain counting** and:

- Starting from a given numerator, we could count *up* any number of steps.

EXAMPLE 7.15.

$$\xrightarrow{21,22,23,...,134,135,136,...}$$

but

- Starting from a given numerator, we could count *down* only a certain number of steps,

EXAMPLE 7.16. Starting from 42, we could count down *at most* 41 steps.

$$\xrightarrow{41,\ 40,\ 39,\ ...\ ,\ 4,\ 3,\ 2,\ 1,\ 0}$$

2. With *signed* numerators, the situation is much simpler: starting from *any* signed numerator, we can count *any* number of steps *up* or *down*.

⧁ ⧀ ⊜ $\textcircled{\leqq}$ $\textcircled{\geqq}$ $\textcircled{\leqq}$ $\textcircled{\neq}$

EXAMPLE 7.17. From $+7$, we can count *up* 17 to $+24$:

$$\xrightarrow{+8,\ +9,\ +10,\ +11,\ \dots\ ,\ +23,\ +24}$$

EXAMPLE 7.18. From $+20$, we can count *down* 7 to $+13$:

$$\xrightarrow{+19,\ +18,\ \dots\ ,\ +14,\ +13}$$

EXAMPLE 7.19. From $+8$, we can count *down* 13 to -5:

$$\xrightarrow{+7,\ +6,\ \dots\ ,\ +2,\ +1,\ 0,\ -1,\ -2,\ -3,\ -4,\ -5}$$

EXAMPLE 7.20. From -7, we can count *up* 13 to $+6$:

$$\xrightarrow{-7,-6,-5,\dots,-2,-1,0,+1,+2,\dots,+5,+6}$$

EXAMPLE 7.21. From -4, we can count *down* 5 to -9:

$$\xrightarrow{+52,\ +51,\ +50,\ +49,\ \dots\ ,\ +2,\ +1,\ 0,\ -1,\ -2,\ -3,\ \dots\ ,\ -22,\ -23}$$

7.5 Comparisons Of Signed Number Phrases

We can compare *signed* number phrases from two different viewpoints depending on whether or not we take the *sign* into consideration:

1. When we compare the signed numerators *themselves*, that is when we *do* take the sign of the numerators into consideration:

- In this text, we will use the same comparison verbs we introduced for plain numerators but inside an "o" as a reminder that we need to take the "o"rientation of the items into consideration:
  ⧁ (ogreater than) ⧀ (oless than) ⊜ (oequal to)
  $\textcircled{\leqq}$ (o greater than or equal) $\textcircled{\geqq}$ (ogreater than or oequal to) $\textcircled{\leqq}$ (oless than or oequal to) $\textcircled{\neq}$ (not oequal to)
- We *count*, up or down as the arrowhead on a signed ruler goes.

EXAMPLE 7.22. In order to go from -5 **Arrows** to $+3$ **Arrows** we must count *up* so we write

$$-5\ \textbf{Arrows} ⧀ +3\ \textbf{Arrows}$$

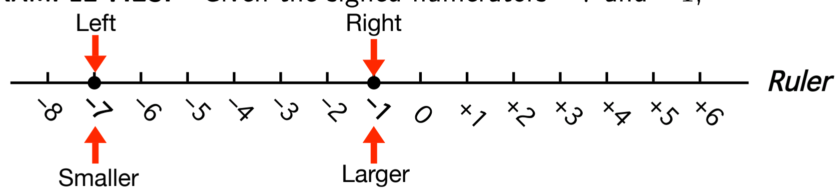
which we read as: Minus FIVE Arrows is oless than Plus THREE Arrows.

- Graphically, regardless of the signs of the two signed numerators,

smaller
larger
closer
farther
size-comparisons

- The **smaller** signed numerator is to the *left*
- The **larger** signed numerator is to the *right*.

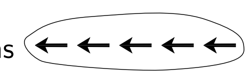
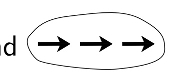
Example 7.23. Given the signed numerators -7 and -1,



and we have $-7 \textcircled{<} -1$

2. Often though, we will only want to know which of the two numerators is **closer** to the *origin* and/or which numerator is **farther** from the origin. Then we must *not* take the *sign* of the numerators into consideration. [1]

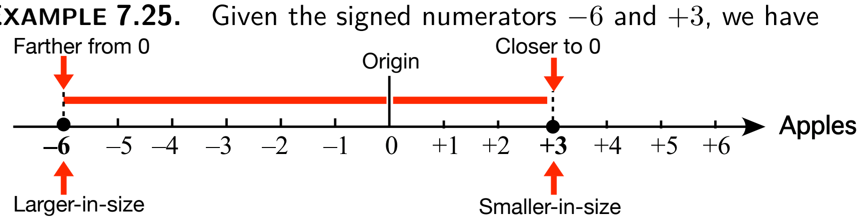
- Unfortunately, there are no symbols for **size-comparisons** and we will just have to write the words.
- We must compare only the *sizes* of the collections, just the way we compared the size of collections of *plain* items.

Example 7.24. Given the collections  and  we write:

-5 **Arrows** *is larger in size than* $+3$ **Arrows**

- Graphically, size-comparison means comparing *how far* from the *origin* the two signed number phrases are.

Example 7.25. Given the signed numerators -6 and $+3$, we have



and since -6 is *farther away from* 0 than $+3$, -6 is *larger in size than* $+3$.

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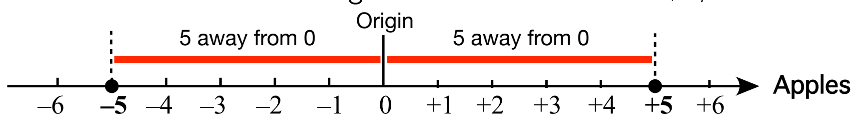
[1] Educologists will surely ask why not use **absolute values**? The answer is that *we* want to compare the *signed* numerators *themselves* while the Educologists would compare something entirely different, namely the *plain* numerators that are their sizes.

boundary
test
ray,␣solid
ray,␣hollow

EXAMPLE 7.26. Given the signed numerators -5 and $+5$, we have



and so, since -5 and $+5$ are equally far from the origin, the signed numerator -5 and $+5$ have the same *size*, namely 5.

7.6 Meeting a Signed Decimal Requirement

When solving comparison problems where the data set is *counting* numerators, we just tried each numerator in the data set by *counting up* or *counting down* to the gauge numerator. With decimal numbers, though, we cannot count up or down.

1. The gauge will be the **boundary** of the solution subset in the sense that all solutions will be on one side or the other of the gauge. What the side is depends of course on the comparison verb but we have only to **test** one numerator on each side of the gauge. Then:

- If the tested numerator is a *solution*, then all the numerators on that side of the *boundary* are also *solutions*.
- If the tested numerator is a *non-solution* then all the numerators on that side of the *boundary* are also *non-solutions*.

2. In order to represent a *solution subset*,

i. We graph the *boundary* of the solution subset exactly the same way as we graphed *counting* number-phrases that is we use

- a *solid dot* ● to graph a boundary that is a *solution*.
- a *hollow dot* ○ to graph a boundary that is a *non-solution*:

ii. • We graph the *solution subset* with a **solid ray**

because this is what we would get if we were to draw a whole lot of *solid dots* right next to each other to graph all the *decimal* numerators that are solutions:

- We graph the part of the data set that is not the *solution subset* with a **hollow ray** because this is what we would get if we were to draw a whole lot of *hollow dots* right next to each other to graph all the *decimal* numerators that are non-solutions:

EXAMPLE 7.27. Given the comparison problem in **Dollars** in which

- the *data set* consists of all possible signed decimal numbers.
- the *comparison formula* is

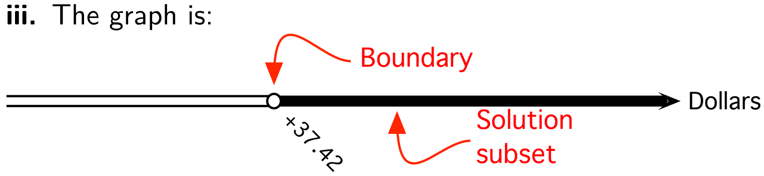
$$x > +37.42$$

i. The *boundary* of the solution subset is $+37.42$ but since $+37.42 > +37.42$ is FALSE, the boundary is a *non-solution*.

ii. To determine on which side of the boundary the *solution subset* is, we pick and test some number on each side of the boundary:

- We pick a numerator to the *left* of $+37.42$, for instance -50, and we try it in the comparison formula:
  $-50 > +37.42$ is FALSE
  so *all* the numerators to the *left* of $+37.42$ are *non-solutions*.
- We pick a numerator to the *right* of $+37.42$, for instance $+100$, and we try it in the comparison formula:
  $+100 > +37.42$ is TRUE
  so *all* the numerators to the *right* of $+37.42$ are *solutions*.

iii. The graph is:



EXAMPLE 7.28. Given the comparison problem in **Dollars** in which

- the *data set* consists of all possible signed decimal numbers.
- the *comparison formula* is

$$x \leqq -358.13$$

i. The *boundary* of the solution subset is -358.13 and since $-358.13 \leqq -358.13$ is TRUE, the boundary is a *solution*.

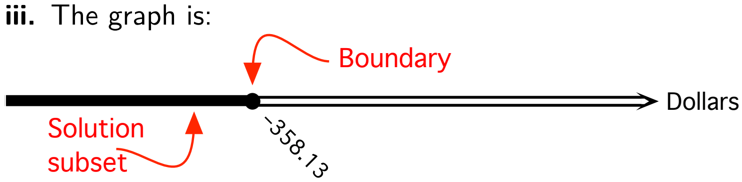
ii. To determine on which side of the boundary the *solution subset* is, we pick and test some number on each side of the boundary:

- We pick a numerator to the *left* of -358.13, for instance -500, and we try it in the comparison formula:

$-500 \leqq -358.13$ is TRUE
so *all* the numerators to the *left* of -358.13 are *solutions*.

- We pick a numerator to the *right* of -358.13, for instance 0, and we try it in the comparison formula:
  $0 \leqq -358.13$ is FALSE
  so *all* the numerators to the *right* of -358.13 are *non-solutions*.

iii. The graph is:



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