Chapter 1

Introduction in Dimension One

Number phrases and Spaces

In order not to lose information when representing on paper real-world collections, we will have to write number phrases, that is phrases made up of a denominator to indicate the nature of the items in the collection and a numerator to indicate the number of items in the collection.

1. Borrowing from the web, we will call a number-phrase that represents a collection of same kind real-world items a basket.

**Example 1.** The number phrase 3 Apples is a basket.

We will call the set of all possible baskets the space of baskets. We can picture the space of baskets with a ruler:

![Ruler diagram]

**Example 2.** The basket 3 Apples can then be pictured as:

![Ruler diagram]

2. Number phrases often come associated with a co-number phrase and baskets often come associated with co-number phrases that correspond to real-world unit-prices and that we will call unit-values.

**Example 3.** A basket such as 3 Apples often comes associated with a unit-value such as 7 Cents Apple (read 7 Cents per Apple.)
We will call the set of all possible unit-values the **space of unit-values**. We can picture the space of unit-values with a ruler:

```
0 1 2 3 4 5 6 7 8 9
```

**Example 4.** The unit-value $7 \frac{\text{Cents}}{\text{Apple}}$ can then be pictured as

```
0 1 2 3 4 5 6 7 8 9
```

**Value and Co-multiplication**

We will call **value** the number phrase that represents the real-world **price** of a given real-world collection of items at a given real-world unit-price.

1. We obtain the value of a basket at a given unit-value from the **co-multiplication** of the basket by the unit-value:
   - The numerator of the value is the product of the numerators,
   - The denominator of the value is the name of the real-world “currency”.

We will use ⊠ as a symbol for co-multiplication.

**Example 5.** $\text{VALUE of 3 Apples at } 7 \frac{\text{Cents}}{\text{Apple}} = 3 \text{ Apples} \times 7 \frac{\text{Cents}}{\text{Apple}} = 21 \text{ Cents}$

2. We will call the set of all possible values the **field of values**. We can picture the field of values also with a ruler:

```
0 5 10 15 20 25 30 35 40 45
```

**Example 6.** The value $21 \text{ Cents}$ can then be pictured as

```
0 5 10 15 20 25 30 35 40 45
```

**Valuation Function**

A **valuation** is a function, specified by a unit-value, that goes from the space of baskets to the field of values that is when we input a basket, a valuation function outputs the value of that basket for the unit-value.

$\text{SPACE of BASKETS} \xrightarrow{\text{VALUATION at unit-value}} \text{FIELD of VALUES}$

The unit-value that specifies the valuation is called the **parameter** of the valuation.
1. In other words, what a valuation function does is to “cost” baskets by outputting the value of these baskets (for the parameter unit-value).

**Example 7.** Given the function $\text{VALUATION}_{\text{at } 7 \text{ Cents per Apple}}$, that is the valuation function specified by the parameter $7 \text{ Cents per Apple}$, when we input the basket $3 \text{ Apples}$, the valuation function outputs the value of the basket $3 \text{ Apples}$ at the unit-value $7 \text{ Cents per Apple}$:

\[
3 \text{ Apples} \xrightarrow{\text{VALUATION}_{\text{at } 7 \text{ Cents per Apple}}} \text{VALUATION}_{\text{at } 7 \text{ Cents per Apple}}(3 \text{ Apples}) = 3 \text{ Apples} \times 7 \text{ Cents per Apple} = 21 \text{ Cents}
\]

2. The graph of an evaluation function

3. The diagram of an evaluation function
4. A reverse problem for a valuation at a given unit-value is therefore, given an amount of money, to determine what basket(s), if any, we can afford at that unit-price for that money.

**Example 8.** Given the unit-value $\frac{7 \text{ Cents}}{\text{Apple}}$, how many Apples can we afford to buy with 21 Cents? In other words, we need to solve:

$$x \text{ Apples} \rightarrow \text{Valuation at } \frac{7 \text{ Cents}}{\text{Apple}}(x \text{ Apples}) = 21 \text{ Cents}$$

We compute the left-hand side:

$$x \text{ Apples} \rightarrow \text{Valuation at } \frac{7 \text{ Cents}}{\text{Apple}}(x \text{ Apples}) = x \text{ Apples} \times \frac{7 \text{ Cents}}{\text{Apple}} = 7x \text{ Cents}$$

so that the reverse problem reduces to solving the equation

$$7x \text{ Cents} = 21 \text{ Cents}$$

that is

$$7x = 21$$

which we can reduce to the basic equation

$$x = 3$$

5. Since the reverse problem always has a unique solution, we can specify an inverse function for the valuation function, that is a function that goes from the field of values back to the space of baskets:

$$\text{FIELD of VALUES} \rightarrow \text{Inverse Valuation at a unit-value} \rightarrow \text{SPACE of BASKETS}$$
and which will output the solution of the reverse problem.

**Example 9.** Since we want that

\[
21 \text{ Cents} \xrightarrow{\text{Inverse-VALUATION at } 7 \text{ Cents/Apple}} \text{Inverse-VALUATION at } 7 \text{ Cents/Apple} (21 \text{ Cents}) = 3 \text{ Apples}
\]

we must set

\[
\# \text{ Cents} \xrightarrow{\text{Inverse-VALUATION at } 7 \text{ Cents/Apple}} \text{Inverse-VALUATION at } 7 \text{ Cents/Apple} (\# \text{ Cents}) = \frac{\# \text{ Cents}}{7 \text{ Cents/Apple}} \times \frac{\text{Apples}}{\# \text{ Cents}} = \frac{\# \text{ Cents}}{7 \text{ Cents/Apple}} \times \frac{\text{Apples}}{\# \text{ Cents}}
\]

**Assessment Function**

An **assessment function** is specified by a basket and goes from the space of unit-values to the field of values, that is, when we input a unit-value, an assessment function outputs the value of the basket at the unit-value.

\[
\text{SPACE of UNIT-VALUES} \xrightarrow{\text{ASSESSMENT on basket}} \text{FIELD of VALUES}
\]

The basket that specifies the assessment function is called the **parameter** of the valuation.

1. In other words, what an assessment function does is to “concretize” unit-values by outputting the values of the parameter basket for these unit-values.

**Example 10.** Given the function \(\text{ASSESSMENT on } 3 \text{ Apples}\) that is the assessment function specified by the parameter \(3 \text{ Apples}\), when we input the unit-value \(7 \text{ Cents/Apple}\), the assessment function outputs the value of the basket \(3 \text{ Apples}\) at the unit-value \(7 \text{ Cents/Apple}\):

\[
7 \text{ Cents/Apple} \xrightarrow{\text{ASSESSMENT on } 3 \text{ Apples}} \text{ASSESSMENT on } 3 \text{ Apples}(7 \text{ Cents/Apple}) = 3 \text{ Apples} \times \frac{7 \text{ Cents/Apple}}{1 \text{ Apple}}
\]

2. The graph of an assessment function
CHAPTER 1. INTRODUCTION IN DIMENSION ONE

3. The diagram of an assessment function

4. A reverse problem for an assessment on a given basket is therefore, given an amount of money, to determine what unit-price(s), if any, we can afford on that basket for that money? In other words, we need to solve:

**Example 11.** Given the basket 3 Apples, how many Cents per Apple can we afford to pay with 21 Cents? In other words, we need to solve:

$$\frac{\text{Cents}}{\text{Apple}} \rightarrow \text{ASSESSMENT on 3 Apples} \left( \frac{\text{Cents}}{\text{Apple}} \right) = 21 \text{ Cents}$$
We compute the left-hand side:

\[
\begin{align*}
\text{Cents} & \quad \text{APPLE} \\
\frac{x}{\text{Apple}} & \quad \text{ASSESSMENT}_{\text{on } 3 \text{ Apples}} \\
\rightarrow & \quad \text{ASSESSMENT}_{\text{on } 3 \text{ Apples}}\left(\frac{x}{\text{Apple}}\right) = 3 \text{ Apples} \otimes x \\
& = 3 \times x \text{ Apples} \\
& = 3x \text{ Cents}
\end{align*}
\]

so that the reverse problem reduces to solving the equation

\[
3x \text{ Cents} = 21 \text{ Cents}
\]

that is

\[
3x = 21
\]

which we can reduce to the basic equation

\[
x = 7
\]

5. Since the reverse problem always has a unique solution, we can specify
an inverse function for the assessment function, that is a function that goes
from the field of values to the space of unit-values.

\[\text{FIELD of VALUES} \quad \text{Inverse ASSESSMENT}_{\text{on basket}} \rightarrow \text{SPACE of UNIT-VALUES}\]

and which will output the solution of the reverse problem.

\textbf{EXAMPLE 12.} Since we want that

\[
21 \text{ Cents} \quad \text{Inverse-ASSESSMENT}_{\text{on 3 Apple}} \rightarrow \text{Inverse-ASSESSMENT}_{\text{on 3 Apple}}(21 \text{ Cents}) = 7 \text{ Cents} \quad \text{Apple}
\]

we must set

\[
\# \text{ Cents} \quad \text{Inverse-ASSESSMENT}_{\text{on 3 Apple}} \rightarrow \text{Inverse-ASSESSMENT}_{\text{on 3 Apple}}(\# \text{ Cents}) = \frac{\# \text{ Cents}}{3 \text{ Apples}} = \frac{\# \text{ Cents}}{3 \text{ Apple}}
\]

So, when we input various VALUES into the function Inverse ASSESSMENT
on 5 apples, the output is the unit value that she gets for that VALUE.
Another way to put it is that what the function Inverse ASSESSMENT on
5 apples does is to ÔstandardizeÔ various VALUES by dividing them by 5
apples.
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