Chapter 1

Introduction in Dimension One

collections
number phrase
denominator
numerator
basket
space of baskets
picture
ruler
co-number phrase
unit-price
unit-value

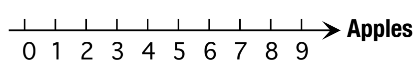
Number phrases and Spaces

In order not to lose information when representing on paper real-world **collections**, we will have to write **number phrases**, that is phrases made up of a **denominator** to indicate the *nature* of the items in the collection and a **numerator** to indicate the *number* of items in the collection.

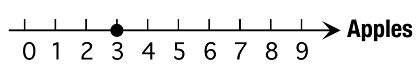
1. Borrowing from the web, we will call a number-phrase that represents a collection of *same kind* real-world items a **basket**.

EXAMPLE 1. The number phrase 3 **Apples** is a basket.

We will call the set of all possible baskets the **space of baskets**. We can **picture** the space of baskets with a **ruler**:



EXAMPLE 2. The basket 3 **Apples** can then be pictured as:

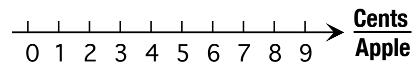


2. Number phrases often come associated with a **co-number phrase** and baskets often come associated with co-number phrases that correspond to real-world **unit-prices** and that we will call **unit-values**.

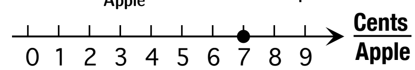
EXAMPLE 3. A basket such as 3 **Apples** often comes associated with a unit-value such as $7\,\frac{\text{Cents}}{\text{Apple}}$ (read 7 **Cents** *per* **Apple**.)

space of unit-values
value
price
co-multiplication
field of values
valuation
parameter

We will call the set of all possible unit-values the **space of unit-values**. We can picture the space of unit-values with a ruler:



EXAMPLE 4. The unit-value $7\,\frac{\textsf{Cents}}{\textsf{Apple}}$ can then be pictured as



Value and Co-multiplication

We will call **value** the number phrase that represents the real-world **price** of a given real-world collection of items at a given real-world unit-price.

1. We obtain the value of a basket at a given unit-value from the **co-multiplication** of the basket by the unit-value:

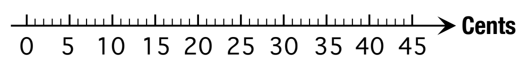
- The numerator of the value is the product of the numerators,
- The denominator of the value is the name of the real-world "currency".

We will use ⊠ as a symbol for co-multiplication.

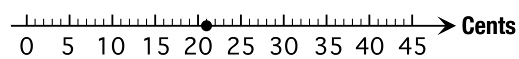
EXAMPLE 5.

$$\begin{aligned} \mathit{VALUE} \text{ of } 3\,\textsf{Apples} \text{ at } 7\,\frac{\textsf{Cents}}{\textsf{Apple}} &= 3\,\textsf{Apples} \boxtimes 7\,\frac{\textsf{Cents}}{\textsf{Apple}} \\ &= 3 \times 7\,\cancel{\textsf{Apples}}\,\frac{\textsf{Cents}}{\cancel{\textsf{Apple}}} \\ &= 21\,\textsf{Cents} \end{aligned}$$

2. We will call the set of all possible values the **field of values**. We can picture the field of values also with a ruler:



EXAMPLE 6. The value $21\,\textsf{Cents}$ can then be pictured as



Valuation Function

A **valuation** is a function, specified by a unit-value, that goes from the space of baskets to the field of values that is when we input a basket, a valuation function outputs the value of that basket for the unit-value.

$$SPACE\,of\,BASKETS \xrightarrow{VALUATION_{at\,unit-value}} FIELD\,of\,VALUES$$

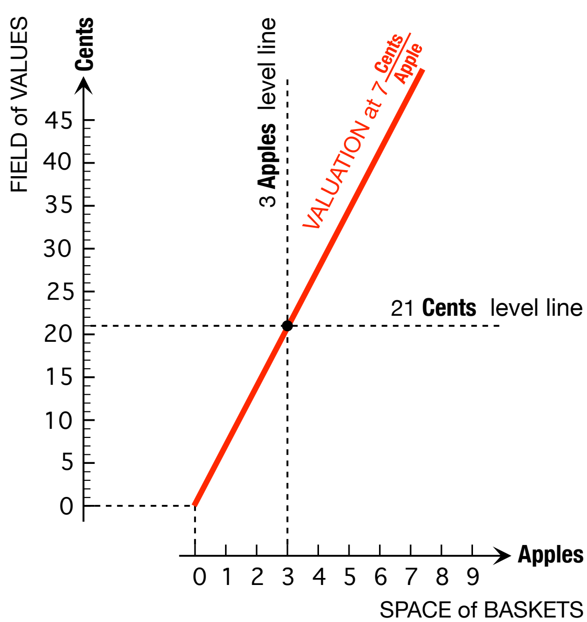
The unit-value that specifies the valuation is called the **parameter** of the valuation.

1. In other words, what a valuation function does is to "cost" baskets by outputting the value of these baskets (for the parameter unit-value). graph diagram

EXAMPLE 7. Given the function $VALUATION_{at\,7\frac{\textsf{Cents}}{\textsf{Apple}}}$, that is the valuation function specified by the parameter $7\,\frac{\textsf{Cents}}{\textsf{Apple}}$, when we input the basket $3\,\textsf{Apples}$, the valuation function outputs the value of the basket $3\,\textsf{Apples}$ at the unit-value $7\frac{\textsf{Cents}}{\textsf{Apple}}$:

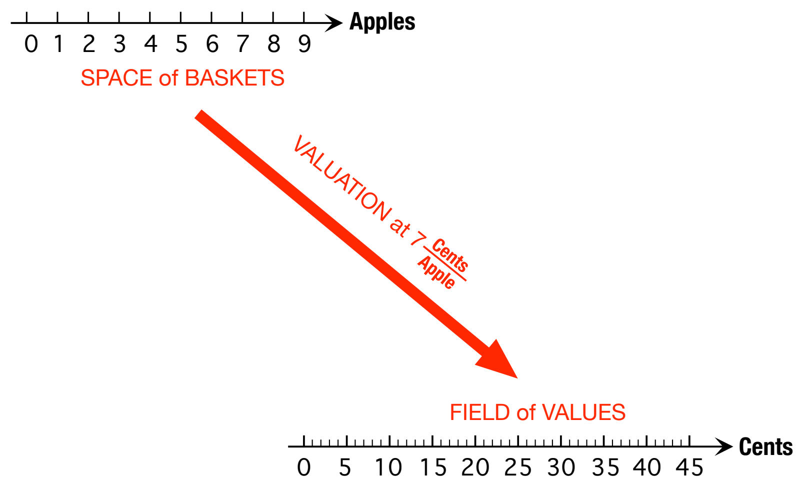
$$3\,\textsf{Apples} \xrightarrow{VALUATION_{at\,7\frac{\textsf{Cents}}{\textsf{Apple}}}} VALUATION_{at\,7\frac{\textsf{Cents}}{\textsf{Apple}}}(3\,\textsf{Apples}) = 3\,\textsf{Apples} \boxtimes 7\frac{\textsf{Cents}}{\textsf{Apple}}$$
$$= 21\,\textsf{Cents}$$

2. The **graph** of an evaluation function



3. The **diagram** of an evaluation function

reverse problem
inverse function



4. A **reverse problem** for a valuation at a given unit-value is therefore, given an amount of money, to determine what basket(s), if any, we can afford at that unit-price for that money.

Example 8. Given the unit-value $7\frac{\textsf{Cents}}{\textsf{Apple}}$, how many **Apples** can we afford to buy with 21 **Cents**? In other words, we need to solve:

$$x\,\textbf{Apples} \xrightarrow{VALUATION_{at\,7\frac{\textsf{Cents}}{\textsf{Apple}}}} VALUATION_{at\,7\frac{\textsf{Cents}}{\textsf{Apple}}}(x\,\textbf{Apples}) = 21\,\textbf{Cents}$$

We compute the left-hand side:

$$\begin{aligned} x\,\textbf{Apples} \xrightarrow{VALUATION_{at\,7\frac{\textsf{Cents}}{\textsf{Apple}}}} VALUATION_{at\,7\frac{\textsf{Cents}}{\textsf{Apple}}}(x\,\textbf{Apples}) &= x\,\textbf{Apples} \boxtimes 7\frac{\textsf{Cents}}{\textsf{Apple}} \\ &= x \times 7\,\cancel{\textbf{Apples}}\,\frac{\textsf{Cents}}{\cancel{\textsf{Apple}}} \\ &= 7x\,\textbf{Cents} \end{aligned}$$

so that the reverse problem reduces to solving the equation

$$7x\,\textbf{Cents} = 21\,\textbf{Cents}$$

that is

$$7x = 21$$

which we can reduce to the basic equation

$$x = 3$$

5. Since the reverse problem always has a unique solution, we can specify an **inverse function** for the valuation function, that is a function that goes from the field of values back to the space of baskets

$$FIELD\,of\,VALUES \xrightarrow{Inverse\,VALUATION_{\text{at a unit-value}}} SPACE\,of\,BASKETS$$

and which will output the solution of the reverse problem. assessment function

EXAMPLE 9. Since we want that

$$21\,\textsf{Cents} \xrightarrow{Inverse\text{-}VALUATION at\, 7\frac{\textsf{Cents}}{\textsf{Apple}}} Inverse\text{-}VALUATION_{at\, 7\frac{\textsf{Cents}}{\textsf{Apple}}}(21\,\textsf{Cents}) = 3\,\textsf{Apples}$$

we must set

$$\begin{aligned}\#\,\textsf{Cents} \xrightarrow{Inverse\text{-}VALUATION_{at\, 7\frac{\textsf{Cents}}{\textsf{Apple}}}} Inverse\text{-}VALUATION_{at\, 7\frac{\textsf{Cents}}{\textsf{Apple}}}(\#\,\textsf{Cents}) &= \frac{\#\,\textsf{Cents}}{7\frac{\textsf{Cents}}{\textsf{Apple}}} \\ &= \frac{\#\,\cancel{\textsf{Cents}}}{7} \boxtimes \frac{\textsf{Apples}}{\cancel{\textsf{Cent}}} \\ &= \frac{\#}{7}\,\textsf{Apples}\end{aligned}$$

Assessment Function

An **assessment function** is specified by a basket and goes from the space of unit-values to the field of values, that is, when we input a unit-value, an assessment function outputs the value of the basket at the unit-value.

$$SPACE\,of\,UNIT\text{-}VALUES \xrightarrow{ASSESSMENT_{on\, basket}} FIELD\,of\,VALUES$$

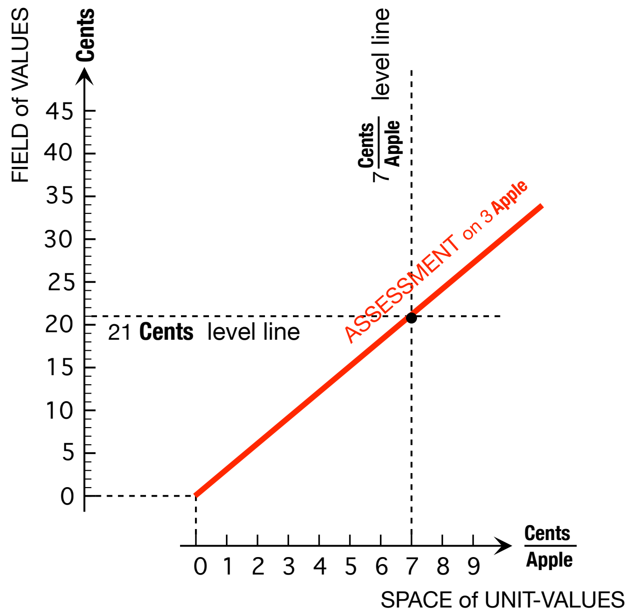
The basket that specifies the assessment function is called the *parameter* of the valuation.

1. In other words, what an assessment function does is to "concretize" unit-values by outputting the values of the parameter basket for these unit-values.

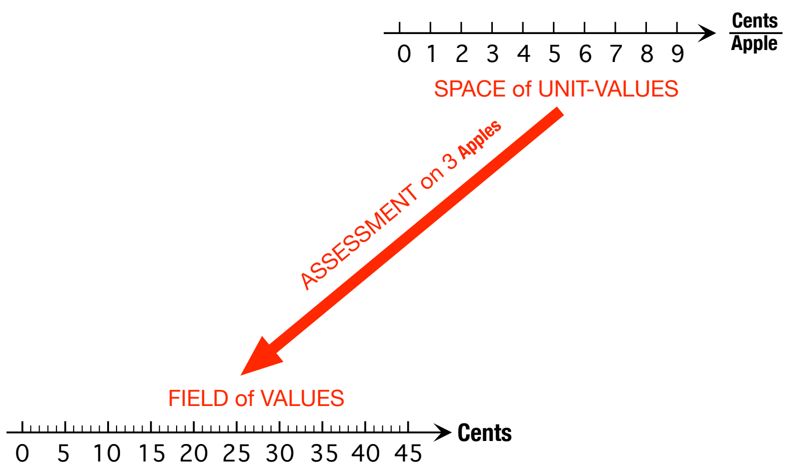
EXAMPLE 10. Given the function $ASSESSMENT_{on\, 3\,\textsf{Apples}}$ that is the assessment function specified by the parameter $3\,\textsf{Apples}$, when we input the unit-value $7\frac{\textsf{Cents}}{\textsf{Apple}}$, the assessment function outputs the value of the basket $3\,\textsf{Apples}$ at the unit-value $7\frac{\textsf{Cents}}{\textsf{Apple}}$:

$$\begin{aligned}7\frac{\textsf{Cents}}{\textsf{Apple}} \xrightarrow{ASSESSMENT_{on\, 3\,\textsf{Apples}}} ASSESSMENT_{on\, 3\,\textsf{Apples}}\left(7\frac{\textsf{Cents}}{\textsf{Apple}}\right) &= 3\,\textsf{Apples} \boxtimes 7\frac{\textsf{Cents}}{\textsf{Apple}} \\ &= 3 \times 7\,\cancel{\textsf{Apples}}\,\frac{\textsf{Cents}}{\cancel{\textsf{Apple}}} \\ &= 21\,\textsf{Cents}\end{aligned}$$

2. The graph of an assessment function



3. The diagram of an assessment function



4. A reverse problem for an assessment on a given basket is therefore, given an amount of money, to determine what unit-price(s), if any, we can afford on that basket for that money? In other words, we need to solve:

Example 11. Given the basket 3 **Apples**. how many **Cents** per **Apple** can we afford to pay with 21 **Cents** In other words, we need to solve:

$$x\,\frac{\textbf{Cents}}{\textbf{Apple}} \xrightarrow{ASSESSMENT_{on\ 3\ \textbf{Apples}}} ASSESSMENT_{on\ 3\ \textbf{Apples}}\left(x\,\frac{\textbf{Cents}}{\textbf{Apple}}\right) = 21\ \textbf{Cents}$$

We compute the left-hand side:

$$x\,\frac{\textsf{Cents}}{\textsf{Apple}} \xrightarrow{ASSESSMENT_{on\,3\,\textsf{Apples}}} ASSESSMENT_{on\,3\,\textsf{Apples}}(x\,\frac{\textsf{Cents}}{\textsf{Apple}}) = 3\,\textsf{Apples} \boxtimes x\,\frac{\textsf{Cents}}{\textsf{Apple}}$$
$$= 3 \times x\,\cancel{\textsf{Apples}}\,\frac{\textsf{Cents}}{\cancel{\textsf{Apple}}}$$
$$= 3x\,\textsf{Cents}$$

so that the reverse problem reduces to solving the equation

$$3x\,\textsf{Cents} = 21\,\textsf{Cents}$$

that is

$$3x = 21$$

which we can reduce to the basic equation

$$x = 7$$

5. Since the reverse problem always has a unique solution, we can specify an inverse function for the assessment function, that is a function that goes from the field of values to the space of unit-values.

$$FIELD\,of\,VALUES \xrightarrow{Inverse\,ASSESSMENT_{on\,basket}} SPACE\,of\,UNIT\text{-}VALUES$$

and which will output the solution of the reverse problem.

Example 12. Since we want that

$$21\,\textsf{Cents} \xrightarrow{Inverse\text{-}ASSESSMENT_{on\,3\,\textsf{Apple}}} Inverse\text{-}ASSESSMENT_{on\,3\,\textsf{Apple}}(21\,\textsf{Cents}) = 7\frac{\textsf{Cents}}{\textsf{Apple}}$$

we must set

$$\#\,\textsf{Cents} \xrightarrow{Inverse\text{-}ASSESSMENT_{on\,3\,\textsf{Apples}}} Inverse\text{-}ASSESSMENT_{on\,3\,\textsf{Apples}}(\#\,\textsf{Cents}) = \frac{\#\,\textsf{Cents}}{3\,\textsf{Apples}}$$
$$= \frac{\#}{3}\,\frac{\textsf{Cents}}{\textsf{Apple}}$$

So, when we input various VALUES into the function Inverse ASSESSMENT on 5 apples, the output is the unit value that she gets for that VALUE. Another way to put it is that what the function Inverse ASSESSMENT on 5 apples does is to ÒstandardizeÓ various VALUES by dividing them by 5 apples.

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