list
co-number phrase
polygon

Chapter 2

Introduction in Dimension Two

As we go from Dimension One to Dimension Two, most everything will remain the same but we *will* encounter one major difference. In fact, the two-dimensional case is a good prototype for any higher dimensions.

Lists and Spaces

Because we will be dealing here with more than one number phrase at a time, we will have to use **lists** of number phrases which we will enclose between square brackets and which we will call *BASKETS*

EXAMPLE 1. The list $[3\,\textsf{Apples}, 17\,\textsf{Bananas}]$ is a *BASKET*.

Just like number phrases often come associated with a **co-number phrase** *BASKETS* often come associated with lists of co-number phrases which we will call *PRICE-LISTS* and which we will write vertically.

EXAMPLE 2. A *BASKET* such as $[3\,\textsf{Apples}, 5\,\textsf{Bananas}]$ could come associated with a *PRICE-LIST* such as $\begin{bmatrix} 7\,\frac{\textsf{Cents}}{\textsf{Apple}} \\ 4\,\frac{\textsf{Cents}}{\textsf{Banana}} \end{bmatrix}$

Then, we will say that

- All possible *BASKETS* make up a *SPACE of BASKETS*

and

- All possible *PRICE-LISTS* make up a *SPACE of PRICE-LISTS*

Finally, in order to picture spaces, we will use **polygons**

co-multiplication
\(\odot\)
evaluation function

Value and Co-multiplication

The cost of a given number of items at a given unit-price is the number phrase whose numerator is the number of items multiplied by the unit-price and whose denominator is the "currency" in the unit-price.

EXAMPLE 3. The *VALUE* of 3 **Apples** at

The *VALUE* of a given *BASKET* at a given *PRICE -LIST* is then given by an operation, called **co-multiplication**, in which we multiply the numerator of the number phrases in the *BASKET* by the numerators of the co-number phrases in the *PRICE -LIST* and add-up. The denominator of the result is the common "currency" in the co-number phrases in the *PRICE-LIST*. We will use the symbol $\odot$ for co-multiplication.

EXAMPLE 4.

$$
\begin{aligned}
VALUE \text{ of } [3\,\textsf{Apples}, 5\,\textsf{Bananas}] \text{ at } \begin{bmatrix} 7\,\frac{\textsf{Cents}}{\textsf{Apple}} \\ 4\,\frac{\textsf{Cents}}{\textsf{Banana}} \end{bmatrix} &= [3\,\textsf{Apples}, 5\,\textsf{Bananas}] \odot \begin{bmatrix} 7\,\frac{\textsf{Cents}}{\textsf{Apple}} \\ 4\,\frac{\textsf{Cents}}{\textsf{Banana}} \end{bmatrix} \\
&= 3\,\cancel{\textsf{Apples}} \times 7\frac{\textsf{Cents}}{\cancel{\textsf{Apple}}} \\
&\quad + 5\,\cancel{\textsf{Bananas}} \times 4\frac{\textsf{Cents}}{\cancel{\textsf{Banana}}} \\
&= 3 \times 7\,\textsf{Cents} + 5 \times 4\,\textsf{Cents} \\
&= 21\,\textsf{Cents} + 20\,\textsf{Cents} \\
&= 41\,\textsf{Cents}
\end{aligned}
$$

Finally, we will call the set of all possible *VALUES* the *FIELD of VALUES* and we have the following picture:

Valuation Function

An **evaluation function** is a function which, given a BASKET, outputs the VALUE of that BASKET for a given PRICE-LIST:

$$BASKET \xrightarrow{EVALUATION|_{PRICE\text{-}LIST}} EVALUATION|_{PRICE\text{-}LIST}\,(BASKET) = VALUE$$

EXAMPLE 5.

$$[3\,\textsf{Apples}, 5\,\textsf{Bananas}] \xrightarrow{VALUATION_{at\begin{bmatrix} 7\,\frac{\textsf{Cents}}{\textsf{Apple}} \\ 4\,\frac{\textsf{Cents}}{\textsf{Banana}} \end{bmatrix}}} VALUATION_{at\begin{bmatrix} 7\,\frac{\textsf{Cents}}{\textsf{Apple}} \\ 4\,\frac{\textsf{Cents}}{\textsf{Banana}} \end{bmatrix}}([3\,\textsf{Apples}, 5\,\textsf{Bananas}]) = 41\,\textsf{Cents}$$

The reverse problem for the function VALUATION is, given a PRICE-LIST, and given a VALUE, to determine what BASKETS, if any, we can get at that VALUE.

EXAMPLE 6.

$$[x\,\textsf{Apples}, y\,\textsf{Bananas}] \xrightarrow{VALUATION_{at\begin{bmatrix} 7\,\frac{\textsf{Cents}}{\textsf{Apple}} \\ 4\,\frac{\textsf{Cents}}{\textsf{Banana}} \end{bmatrix}}} VALUATION_{at\begin{bmatrix} 7\,\frac{\textsf{Cents}}{\textsf{Apple}} \\ 4\,\frac{\textsf{Cents}}{\textsf{Banana}} \end{bmatrix}}([x\,\textsf{Apples}, y\,\textsf{Bananas}]) = 41\,\textsf{Cents}$$

We have:

$$\begin{aligned} VALUATION_{at\begin{bmatrix} 7\,\frac{\textsf{Cents}}{\textsf{Apple}} \\ 4\,\frac{\textsf{Cents}}{\textsf{Banana}} \end{bmatrix}}([x\,\textsf{Apples}, y\,\textsf{Bananas}]) &= [x\,\textsf{Apples}, y\,\textsf{Bananas}] \odot \begin{bmatrix} 7\,\frac{\textsf{Cents}}{\textsf{Apple}} \\ 4\,\frac{\textsf{Cents}}{\textsf{Banana}} \end{bmatrix} \\ &= x \cdot 7\,\textsf{Cents} + y \cdot 4\,\textsf{Cents} \\ &= 41\,\textsf{Cents} \end{aligned}$$

So the BASKETS that we can get with $41\,\textsf{Cents}$ are given by the solutions of the equation

$$7x + 4y = 41$$

Assessment Function